UNIT – I

Fundamentals of Vibration Analysis

Introduction : Vibrations occur in many aspects of our life. For example, in the human body, there are low-frequency oscillations of the lungs and the heart, high-frequency oscillations of the ear, oscillations of the larynx as one speaks, and oscillations induced by rhythmical body motions such as walking, jumping, and dancing.

Many man-made systems also experience or produce vibrations. For example, any unbalance in machines with rotating parts such as fans, ventilators, centrifugal separators, washing machines, lathes, centrifugal pumps, rotary presses, and turbines, can cause vibrations. For these machines, vibrations are generally undesirable. Buildings and structures can experience vibrations due to operating machinery; passing vehicular, air, and rail traffic; or natural phenomena such as earthquakes and winds. Pedestrian bridges and floors in buildings also experience vibrations due to human movement on them. In structural systems, the fluctuating stresses due to vibrations can result in fatigue failure. Vibrations are also undesirable when performing measurements with precision instruments such as an electron microscope and when abricating micro electro mechanical systems. In vehicle design, noise due to vibrating panels must be reduced. Vibrations, which can be responsible for unpleasant sounds called noise, are also responsible for the music that we hear. Vibrations are also beneficial for many purposes such as atomic clocks that are based on atomic vibrations, vibratory parts feeders, paint mixers, ultrasonic instrumentation used in eye and other types of surgeries, sirens and alarms for warnings, determination of fundamental properties of thin films from an understanding of atomic vibrations, and stimulation of bone growth.

It is likely that the early interest in vibrations was due to development of musical instruments such as whistles and drums. As early as 4000 B.C., it is believed that in India and China there was an interest in understanding music, which is described as a pulsating effect due to rapid change in pitch. The origin of the harmonica can be traced back to 3000 B.C., when in China, a bamboo reed instrument called a "sheng" was introduced. From archeological studies of the royal tombs in Egypt, it is known that stringed instruments have also been around from about 3000 B.C. A first scientific study into such instruments is attributed to the Greek philosopher and mathematician Pythagoras (582–507 B.C.). He showed that if two like strings are subjected to equal tension, and if one is half the length of the other, the tones they produce are an octave (a factor of two) apart. It is interesting to note that although music is considered a highly subjective and personal art, it is closely governed by vibration principles such as those determined by Pythagoras and others who followed him.

The vibrating string was also studied by Galileo Galilei (1564–1642), who was the first to show that pitch is related to the frequency of vibration. Galileo also laid the foundations for studies of vibrating systems through his observations made in 1583 regarding the motions of a lamp hanging from a cathedral in Pisa, Italy. He found that the period of motion was independent of the amplitude of the swing of the lamp. This property holds for all vibratory systems that can be described by linear models. The pendulum system studied by Galileo has been used as a paradigm to illustrate the principles of vibrations for many centuries. Galileo and many others who followed him have laid the foundations for vibrations, which is a discipline that is generally grouped under the umbrella of mechanics.

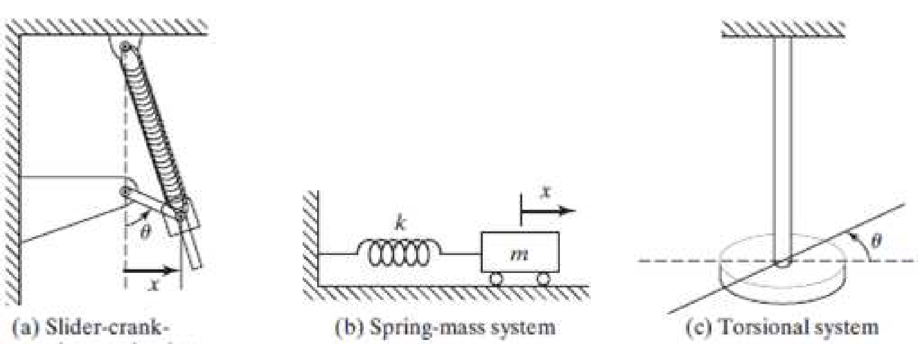
Importance of the Study of Vibration:

Most human activities involve vibration in one form or other. For example, we hear because our eardrums vibrate and see because light waves undergo vibration. Breathing is associated with the vibration of lungs and walking involves (periodic) oscillatory motion of legs and hands. Human speech requires the oscillatory motion of larynges (and tongues) [1.17]. Early scholars in the field of vibration concentrated their efforts on understanding the natural phenomena and developing mathematical theories to describe the vibration of physical systems. In recent times, many investigations have been motivated by the engineering applications of vibration, such as the design of machines, foundations, structures, engines, turbines, and control systems. Most prime movers have vibrational problems due to the inherent unbalance in the engines. The unbalance may be due to faulty design or poor manufacture. Imbalance in diesel engines, for example, can cause ground waves sufficiently powerful to create a nuisance in urban areas. The wheels of some locomotives can rise more than a centimeter off the track at high speeds due to imbalance. In turbines, vibrations cause spectacular mechanical failures. Engineers have not yet been able to prevent the failures that result from blade and disk vibrations in turbines. Naturally, the structures designed to support heavy centrifugal machines, like motors and turbines, or reciprocating machines, like steam and gas engines and reciprocating pumps, are also subjected to vibration. In all these situations, the structure or machine component subjected to vibration can fail because of material fatigue resulting from the cyclic variation of the induced stress. Furthermore, the vibration causes more rapid wear of machine parts such as bearings and gears and also creates excessive noise. In machines, vibration can loosen fasteners such as nuts. In metal cutting processes, vibration can cause chatter, which leads to a poor surface finish. Whenever the natural frequency of vibration of a machine or structure coincides with the frequency of the external excitation, there occurs a phenomenon known as *resonance*, which leads to excessive deflections and failure. The literature is full of accounts of system failures brought about by resonance and excessive vibration of components and systems Because of the devastating effects that vibrations can have on machines and structures, vibration testing has become a standard procedure in the design and development of most engineering systems .

In many engineering systems, a human being acts as an integral part of the system. The transmission of vibration to human beings results in discomfort and loss of efficiency. The vibration and noise generated by engines causes annoyance to people and, sometimes, damage to property. Vibration of instrument panels can cause their malfunction or difficulty in reading the meters. Thus one of the important purposes of vibration study is to reduce vibration through proper design of machines and their mountings. In this connection, the mechanical engineer tries to design the engine or machine so as to minimize imbalance, while the structural engineer tries to design the supporting structure so as to ensure that the effect of the imbalance will not be harmful .

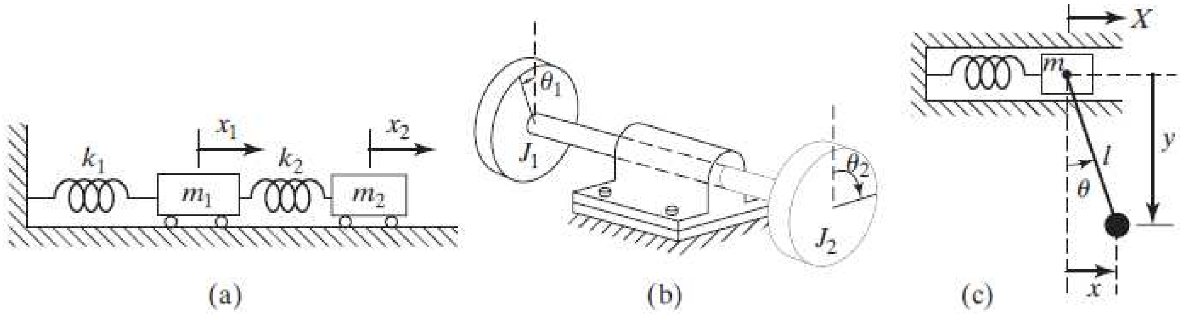
Number of Degrees of Freedom:

The minimum number of independent coordinates required to determine completely the positions of all parts of a system at any instant of time defines the number of degrees of freedom of the system. The simple pendulum shown in Fig. , as well as each of the systems shown in Fig. , represents a single-degree-of-freedom system. For example, the motion of the simple pendulum can be stated either in terms of the angle or in terms of the Cartesian coordinates x and y. If the coordinates x and y are used to describe the motion, it must be recognized that these coordinates are not independent.

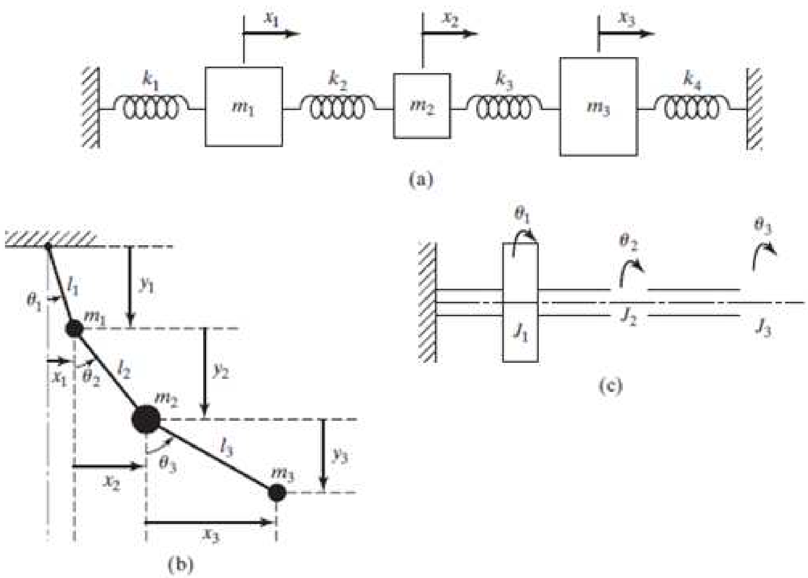


(a) Slider-crank-spring mechanism (b) Spring-mass system (c) Torsional system

Single-degree-of-freedom systems.



Two-degree-of-freedom systems.



Three–degree-of-freedom systems.

Discrete and Continuous Systems:

A large number of practical systems can be described using a finite number of degrees of freedom. Some systems, especially those involving continuous elastic members, have an infinite number of degrees of freedom. As a simple example, consider the cantilever beam . Since the beam has an infinite number of mass points, we need an infinite number of coordinates to specify its deflected configuration. The infinite number of coordinates defines its elastic deflection curve. Thus the cantilever beam has an infinite number of degrees of freedom.

Most structural and machine systems have deformable (elastic) members and therefore have an infinite number of degrees of freedom. Systems with a finite number of degrees of freedom are called *discrete* or *lumped parameter* systems, and those with an infinite number of degrees of freedom are called *continuous* or *distributed* systems. Most of the time, continuous systems are approximated as discrete systems, and solutions are obtained in a simpler manner. Although treatment of a system as continuous gives exact results, the analytical methods available for dealing with continuous systems are limited to a narrow selection of problems, such as uniform beams, slender rods, and thin plates. Hence most of the practical systems are studied by treating them as finite lumped masses, springs, and dampers. In general, more accurate results are obtained by increasing the number of masses, springs, and dampers that is, by increasing the number of degrees of freedom

Types of Vibrations :

Longitudinal vibrations: When the particles of the shaft or disc moves parallel to the axis of the shaft, as shown in Fig. (a), then the vibrations are known as longitudinal vibrations. In this case, the shaft is elongated and shortened alternately and thus the tensile and compressive stresses are induced alternately in the shaft.

Transverse vibrations: When the particles of the shaft or disc move approximately perpendicular to the axis of the shaft, as shown in Fig. (b), then the vibrations are known as transverse vibrations. In this case, the shaft is straight and bent alternately and bending stresses are induced in the shaft.

Torsional vibrations: When the particles of the shaft or disc move in a circle about the axis of the shaft, as shown in Fig. (c), then the vibrations are known as torsional vibrations. In this case, the shaft is twisted and untwisted alternately and the torsional shear stresses are induced in the shaft.

Free Vibration. If a system, after an initial disturbance, is left to vibrate on its own, the ensuing vibration is known as *free vibration*. No external force acts on the system. The oscillation of a simple pendulum is an example of free vibration.

Forced Vibration. If a system is subjected to an external force (often, a repeating type of force), the resulting vibration is known as *forced vibration*. The oscillation that arises in machines such as diesel engines is an example of forced vibration.

If the frequency of the external force coincides with one of the natural frequencies of the system, a condition known as *resonance* occurs, and the system undergoes dangerously large oscillations. Failures of such structures as buildings, bridges, turbines, and airplane wings have been associated with the occurrence of resonance.

Undamped and Damped Vibration

If no energy is lost or dissipated in friction or other resistance during oscillation, the vibration is known as *undamped vibration*. If any energy is lost in this way, however, it is called *damped vibration*. In many physical systems, the amount of damping is so small that it can be disregarded for most engineering purposes. However, consideration of damping becomes extremely important in analyzing vibratory systems near resonance

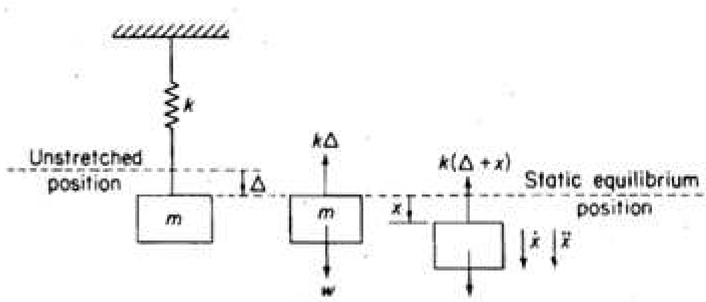
Linear and Nonlinear Vibration

If all the basic components of a vibratory system the spring, the mass, and the damper behave linearly, the resulting vibration is known as *linear vibration.* If, however, any of the basic components behave nonlinearly, the vibration is called *nonlinear vibration.* The differential equations that govern the behavior of linear and nonlinear vibratory systems are linear and nonlinear, respectively. If the vibration is linear, the principle of superposition holds, and the mathematical techniques of analysis are well developed. For nonlinear vibration, the superposition principle is not valid, and techniques of analysis are less well known. Since all vibratory systems tend to behave nonlinearly with increasing amplitude of oscillation, a knowledge of nonlinear vibration is desirable in dealing with practical vibratory systems.

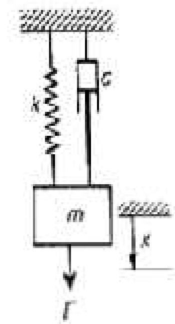
Deterministic and Random Vibration

If the value or magnitude of the excitation (force or motion) acting on a vibratory system is known at any given time, the excitation is called *deterministic.* The resulting vibration is known as *deterministic vibration.*

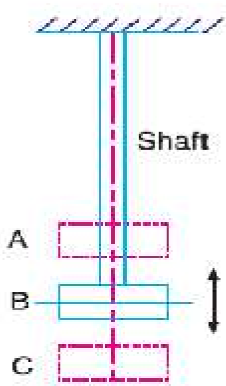
In some cases, the excitation is *nondeterministic* or *random;* the value of the excitation at a given time cannot be predicted. In these cases, a large collection of records of the excitation may exhibit some statistical regularity. It is possible to estimate averages such as the mean and mean square values of the excitation. Examples of random excitations are wind velocity, road roughness, and ground motion during earthquakes. If the excitation is random, the resulting vibration is called *random vibration.* In this case the vibratory response of the system is also random; it can be described only in terms of statistical quantities.



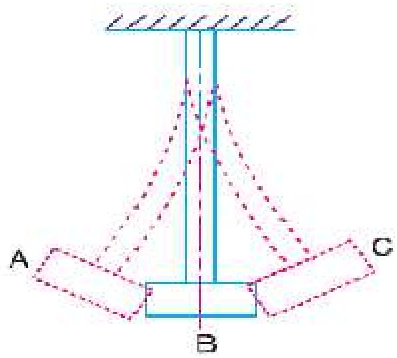
Free Vibrations



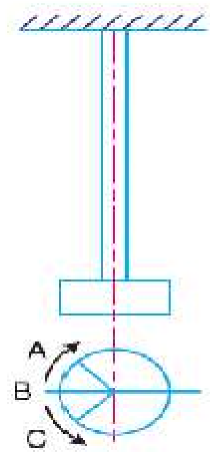
Forced Vibrations



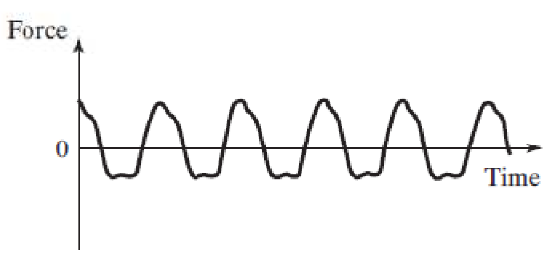
Longitudinal Vibrations



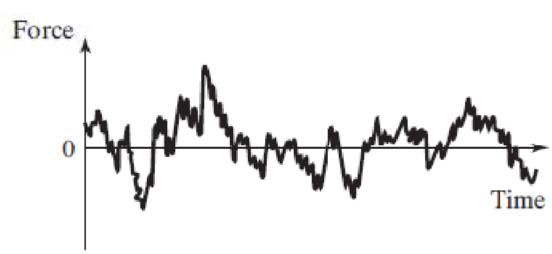
Transverse Vibrations



Torsional Vibrations



(a) A deterministic (periodic) excitation



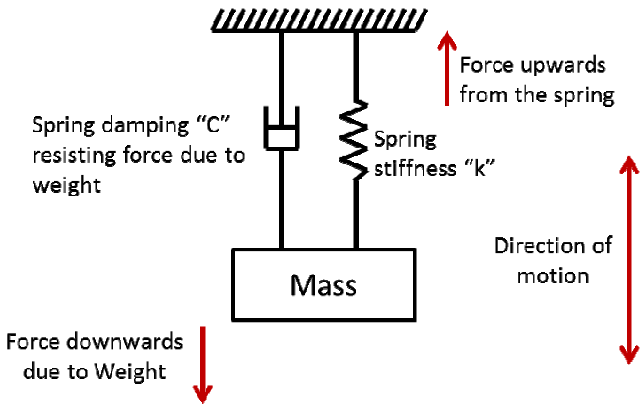
(b) A random excitation

Elements of Vibrating system :

There are, in general, three elements that comprise a vibrating system:

i) inertia elements,
ii) stiffness elements, and
iii) dissipation elements.

In addition to these elements, one must also consider externally applied forces and moments and external disturbances from prescribed initial displacements and/or initial velocities.

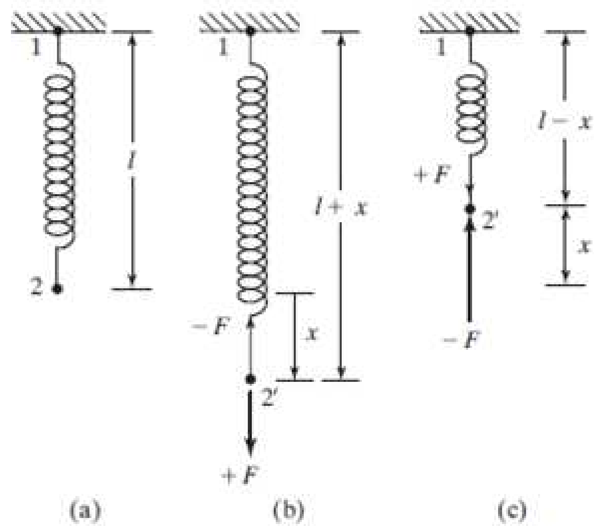


The inertia element stores and releases kinetic energy, the stiffness element stores and releases potential energy, and the dissipation or damping element is used to express energy loss in a system. Each of these elements has different excitation-response characteristics and the excitation is in the form of either a force or a moment and the corresponding response of the element is in the form of a displacement, velocity, or acceleration. The inertia elements are characterized by a relationship between an applied force (or moment) and the corresponding acceleration response. The stiffness elements are characterized by a relationship between an applied force (or moment) and the corresponding displacement (or rotation) response. The dissipation elements are characterized by a relationship between an applied force (or moment) and the corresponding velocity response.

Quantity	Units
Translational motion	
Mass, m	kg
Stiffness, k	N/m
Damping, c	N·s/m
External force, F	N
Rotational motion	
Mass moment of inertia, J	kg·m^2
Stiffness, k_t	N·m/rad
Damping, c_t	N·m·s/rad
External moment, M	N·m

Spring Elements

A spring is a type of mechanical link, which in most applications is assumed to have negligible mass and damping. The most common type of spring is the helical-coil spring used in retractable pens and pencils, staplers, and suspensions of freight trucks and other vehicles. Several other types of springs can be identified in engineering applications. In fact, any elastic or deformable body or member, such as a cable, bar, beam, shaft or plate, can be considered as a spring. A spring is commonly represented as shown in Fig. . If the free length of the spring, with no forces acting, is denoted *l*, it undergoes a change in length when an axial force is applied. For example, when a tensile force *F* is applied at its free end 2, the spring undergoes an elongation *x* as shown in Fig. , while a compressive force *F* applied at the free end 2 causes a reduction in length *x* as shown in Fig. .



Deformation of a spring.

A spring is said to be linear if the elongation or reduction in length *x* is related to the applied force *F* as

$$F=kx$$

where *k* is a constant, known as the *spring constant* or *spring stiffness* or *spring rate*. The spring constant *k* is always positive and denotes the force (positive or negative) required to cause a unit deflection longation or reduction in length) in the spring. When the spring is stretched (or compressed) under a tensile (or compressive) force *F*, according to Newton s third law of motion, a restoring force or reaction of magnitude +F or –F is developed opposite to the applied force. This restoring force tries to bring the stretched (or compressed) spring back to its original unstretched or free length as shown in Fig. (or 1.18(c)). If we plot a graph between *F* and *x*, the result is a straight line according to Eq. (1.1). The work done (*U*) in deforming a spring is stored as strain or potential energy in the spring, and it is given by

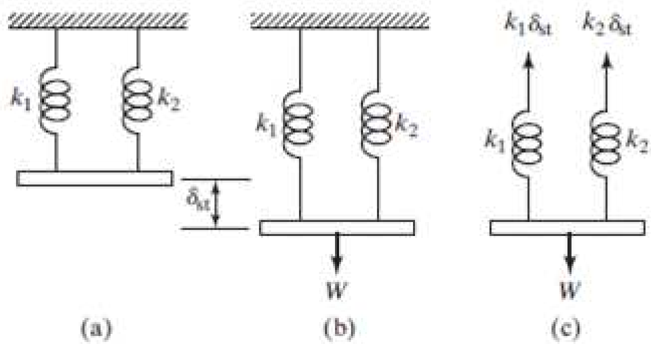
$$U = \frac{1}{2}kx^2$$

Combination of Springs

In many practical applications, several linear springs are used in combination. These springs can be combined into a single equivalent spring as indicated below

Case 1: Springs in Parallel. To derive an expression for the equivalent spring constant of springs connected in parallel, consider the two springs shown in Fig. . When a load *W* is applied, the system undergoes a static deflection as shown in Fig. (b). Then the free-body diagram, shown in Fig. (c), gives the equilibrium equation

$$W = k_1\delta_{st} + k_2\delta_{st}$$



Springs in parallel.

If k_{eq} denotes the equivalent spring constant of the combination of the two springs, then for the same static deflection δ_{st}, we have

$$W = k_{eq}\delta_{st}$$

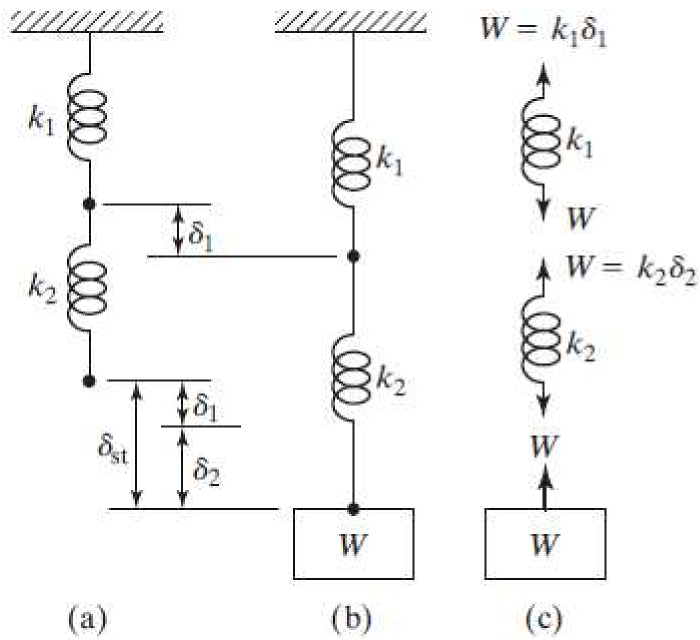
Equations (1.8) and (1.9) give

$$k_{eq} = k_1 + k_2$$

In general, if we have n springs with spring constants $k_1, k_2, \ldots, k_n$ in parallel, then the equivalent spring constant k_{eq} can be obtained:

$$k_{eq} = k_1 + k_2 + \cdots + k_n$$

Case 2: Springs in Series. Next we derive an expression for the equivalent spring constant of springs connected in series by considering the two springs shown in Fig. (a). Under the action of a load *W*, springs 1 and 2 undergo elongations and respectively, as shown in Fig. (b).



The total elongation (or static deflection) of the system, is given by

$$\delta_{st} = \delta_1 + \delta_2$$

Since both springs are subjected to the same force W, we have the equilibrium shown in Fig. (c):

$$W = k_1\delta_1$$
$$W = k_2\delta_2$$

If k_{eq} denotes the equivalent spring constant, then for the same static deflection,

$$W = k_{eq}\delta_{st}$$

$$k_1\delta_1 = k_2\delta_2 = k_{eq}\delta_{st}$$

or

$$\delta_1 = \frac{k_{eq}\delta_{st}}{k_1} \quad \text{and} \quad \delta_2 = \frac{k_{eq}\delta_{st}}{k_2}$$

Substituting these values of δ_1 and δ_2 into Eq. (1.12), we obtain

$$\frac{k_{eq}\delta_{st}}{k_1} + \frac{k_{eq}\delta_{st}}{k_2} = \delta_{st}$$

—that is,

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

Equation (1.16) can be generalized to the case of n springs in series:

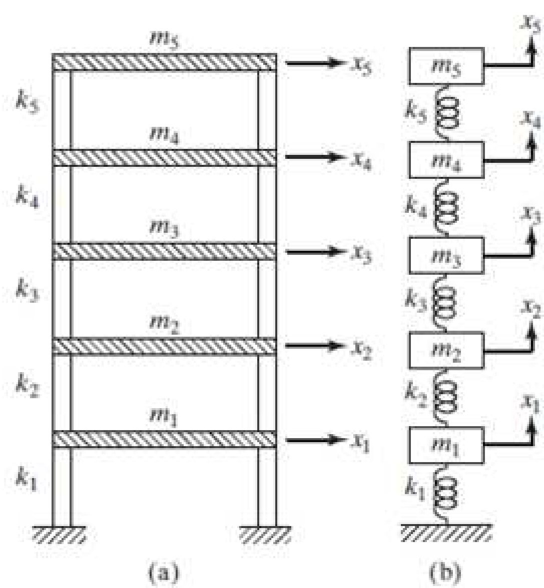
$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}$$

In certain applications, springs are connected to rigid components such as pulleys, levers, and gears. In such cases, an equivalent spring constant can be found using energy equivalence

Mass or Inertia Elements

The mass or inertia element is assumed to be a rigid body; it can gain or lose kinetic energy whenever the velocity of the body changes. From Newton s second law of motion, the product of the mass and its acceleration is equal to the force applied to the mass. Work is equal to the force multiplied by the displacement in the direction of the force, and the work done on a mass is stored in the form of the mass s kinetic energy. In most cases, we must use a mathematical model to represent the actual vibrating system, and there are often several possible models. The purpose of the analysis often determines which mathematical model is appropriate. Once the model is chosen, the mass or inertia elements of the system can be easily identified.

In many practical applications, several masses appear in combination. For a simple analysis, we can replace these masses by a single equivalent mass



Idealization of a multistory building as a multi-degree-of-freedom system.

Damping Elements

In many practical systems, the vibrational energy is gradually converted to heat or sound. Due to the reduction in the energy, the response, such as the displacement of the system, gradually decreases. The mechanism by which the vibrational energy is gradually converted into heat or sound is known as *damping*. Although the amount of energy converted into heat or sound is relatively small, the consideration of damping becomes important for an accurate prediction of the vibration response of a system. A damper is assumed to have neither mass nor elasticity, and damping force exists only if there is relative velocity between the two ends of the damper. It is difficult to determine the causes of damping in practical systems. Hence damping is modeled as one or more of the following types.

Viscous Damping. Viscous damping is the most commonly used damping mechanism in vibration analysis. When mechanical systems vibrate in a fluid medium such as air, gas, water, or oil, the resistance offered by the fluid to the moving body causes energy to be dissipated. In this case, the amount of dissipated energy depends on many factors, such as the size and shape of the vibrating body, the viscosity of the fluid, the frequency of vibration, and the velocity of the vibrating body. In viscous damping, the damping force is proportional to the velocity of the vibrating body Typical examples of viscous damping include (1) fluid film between sliding surfaces, (2) fluid flow around a piston in a cylinder, (3) fluid flow through an orifice, and (4) fluid film around a journal in a bearing.

Coulomb or Dry-Friction Damping. Here the damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating body. It is caused by friction between rubbing surfaces that either are dry or have insufficient lubrication.

Material or Solid or Hysteretic Damping. When a material is deformed, energy is absorbed and dissipated by the material . The effect is due to friction between the internal planes, which slip or slide as the deformations take place. When a body having material damping is subjected to vibration, the stress-strain diagram shows a hysteresis loop as indicated . The area of this loop denotes the energy lost per unit volume of the body per cycle due to damping.

Vibration Analysis Procedure

A vibratory system is a dynamic one for which the variables such as the excitations (inputs) and responses (outputs) are time dependent. The response of a vibrating system generally depends on the initial conditions as well as the external excitations. Most practical vibrating systems are very complex, and it is impossible to consider all the details for a mathematical analysis. Only the most important features are considered in the analysis to predict the behavior of the system under specified input conditions. Often the overall behavior of the system can be determined by considering even a simple model of the complex physical system. Thus the analysis of a vibrating system usually involves mathematical modeling, derivation of the governing equations, solution of the equations, and interpretation of the results.

Step 1: Mathematical Modeling. The purpose of mathematical modeling is to represent all the important features of the system for the purpose of deriving the mathematical (or analytical) equations governing the system s behavior. The mathematical model should include enough details to allow describing the system in terms of equations without making it too complex. The mathematical model may be linear or nonlinear, depending on the behavior of the system s components. Linear models permit quick solutions and are simple to handle; however, nonlinear models sometimes reveal certain characteristics of the system that cannot be predicted using linear models

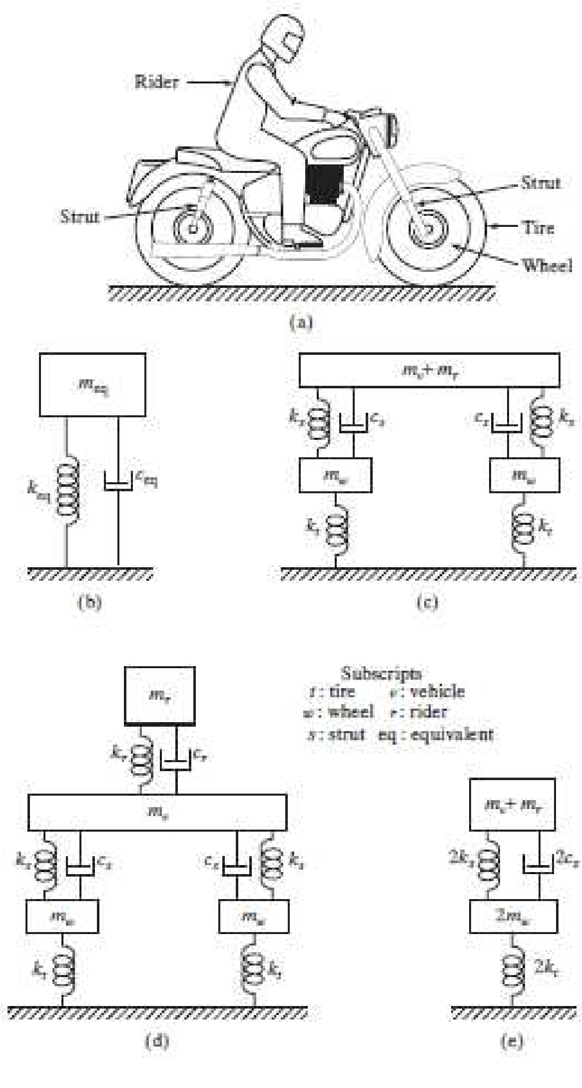
Step 2: Derivation of Governing Equations. Once the mathematical model is available, we use the principles of dynamics and derive the equations that describe the vibration of the system. The equations of motion can be derived conveniently by drawing the free-body diagrams of all the masses involved. The free-body diagram of a mass can be obtained by isolating the mass and indicating all externally applied forces, the reactive forces, and the inertia forces. The equations of motion of a vibrating system are usually in the form of a set of ordinary differential equations for a discrete system and partial differential equations for a continuous system. The equations may be linear or nonlinear, depending on the behavior of the components of the system. Several approaches are commonly used to derive the governing equations. Among them are Newton s second law of motion, D Alembert s principle, and the principle of conservation of energy.

Step 3: Solution of the Governing Equations. The equations of motion must be solved to find the response of the vibrating system. Depending on the nature of the problem, we can use one of the following techniques for finding the solution: standard methods of solving differential equations, Laplace transform methods, matrix methods,1 and numerical methods. If the governing equations are nonlinear, they can seldom be solved in closed form. Furthermore, the solution of partial differential equations is far more involved than that of ordinary differential equations. Numerical methods involving computers can be used to solve the equations.

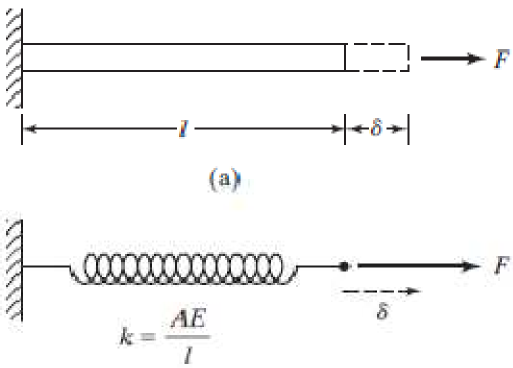
Step 4: Interpretation of the Results. The solution of the governing equations gives the displacements, velocities, and accelerations of the various masses of the system. These results must be interpreted with a clear view of the purpose of the analysis and the possible design implications of the results.

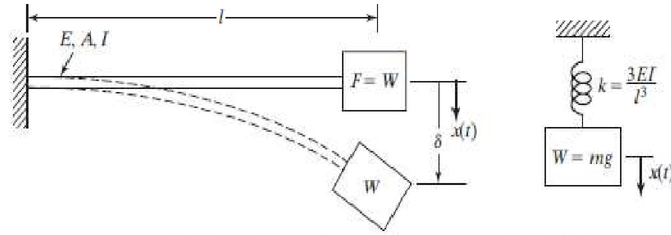
Examples :

Mathematical Model of a Motorcycle



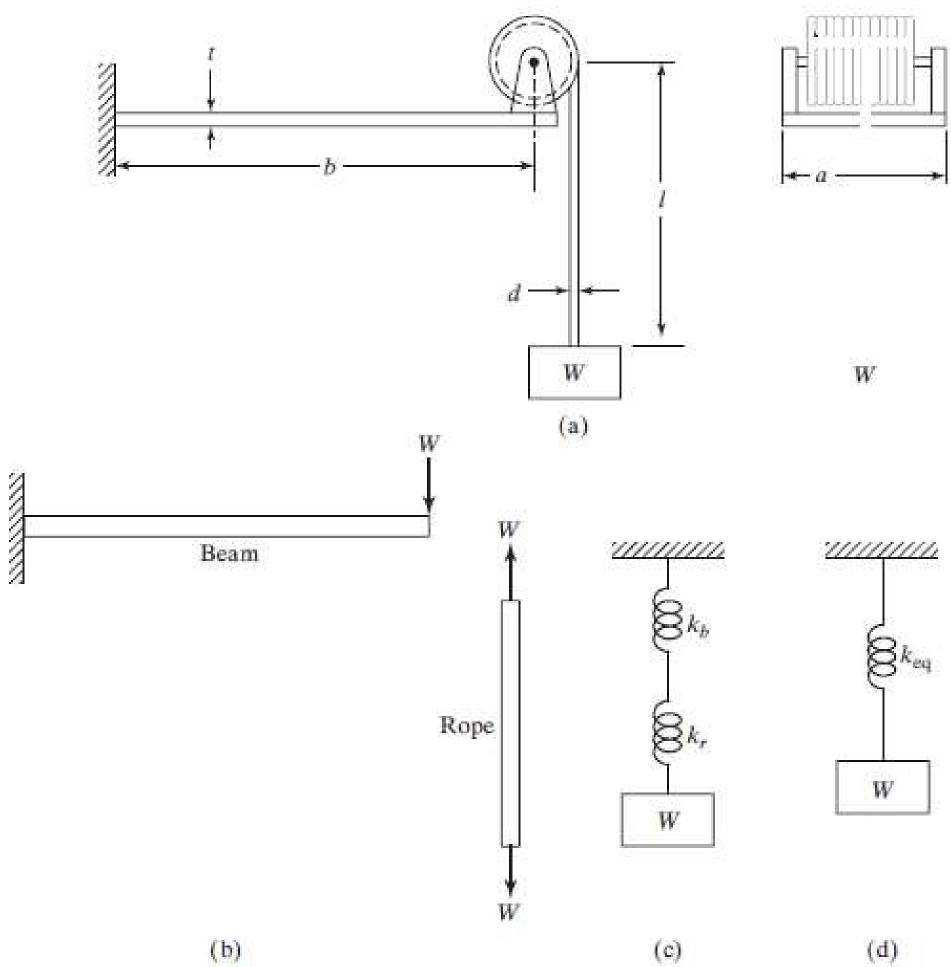
Structural bar and beam





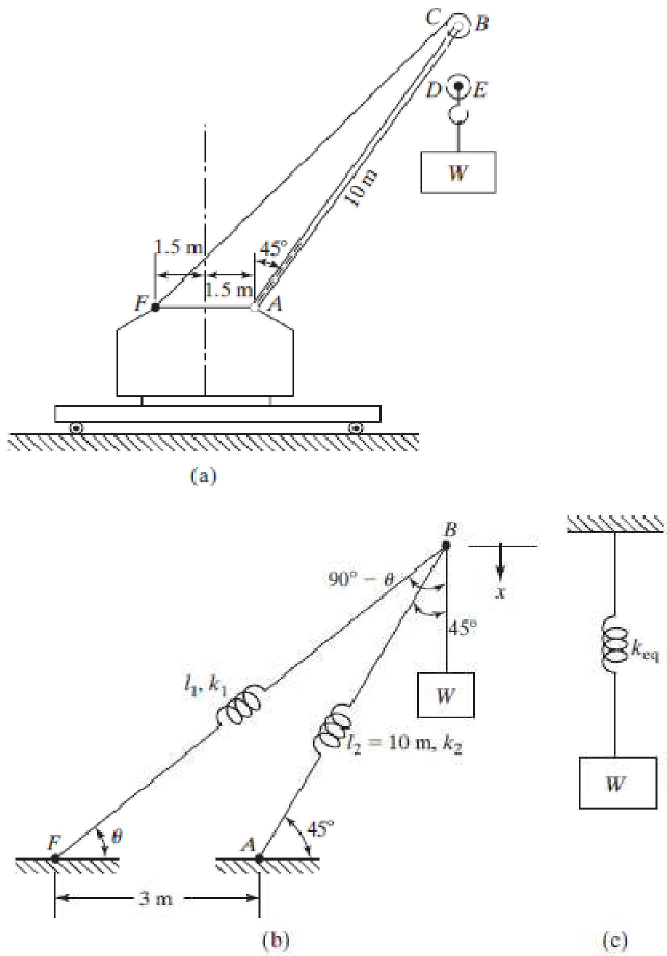
(a) Cantilever with end force (b) Equivalent spring

Hoisting Drum :



Hoisting drum.

Crane :



C
B
D
E
W
10 m
1.5 m
45°
1.5 m
F
A
(a)
B
90° − θ
x
45°
l_1, k_1
W
l_2 = 10 m, k_2
F
θ
A
45°
3 m
(b)
k_{eq}
W
(c)

MECHANICAL VIBRATIONS

UNIT II

Free Vibrations of Single Degree of Freedom Systems

Undamped Free Vibrations: Governing differential equation , Newton's method , Energy method, Rayleigh's method, torsional system – equations of motion and solution.

Damped Vibrations: Governing differential equation, critical damping coefficient and damping ratio, damped natural frequency, logarithmic decrement, energy dissipated in viscous damping.

A practical system is very complicated. Therefore, before proceeding to analyse the system it is desirable to simplify it by modeling the system. The modeling of the system is carried over in such a manner that the result is acceptable within the desirable accuracy. Instead of considering distributed mass, a lumped mass is easier to analyse, whose dynamic behaviour can be determined by one independent principal coordinate, in a single degree freedom system. It is important to study the single degree freedom system for a clear understanding of basic features of a vibration problem.

Elements of Lumped Parameter Vibratory System

The elements constituting a lumped parameter vibratory system are :

The Mass

The mass is assumed to be rigid and concentrated at the centre of gravity.

The Spring

It is assumed that the elasticity is represented by a helical spring. When deformed it stores energy. The energy stored in the spring is given by

$$PE = \frac{1}{2} k x^2$$

where k is stiffness of the spring. The force at the spring is given by

$$F = k x$$

The springs work as energy restoring element. They are treated massless.

The Damper

In a vibratory system the damper is an element which is responsible for loss of energy in the system. It converts energy into heat due to friction which may be either sliding friction or viscous friction. A vibratory system stops vibration because of energy conversion by damper. There are two types of dampers.

Viscous Damper

A viscous damper consists of viscous friction which converts energy into heat due to this. For this damper, force is proportional to the relative velocity.

$$F_d \;\alpha\; \text{relative velocity } (v)$$

$$\therefore \qquad F_d = c\,v$$

where c is constant of proportionality and it is called coefficient of damping.

The coefficient of viscous damping is defined as the force in 'N' when velocity is 1 m/s.

Coulumb's Damper

The dry sliding friction acts as a damper. It is almost a constant force but direction is always opposite to the sliding velocity. Therefore, direction of friction changes due to change in direction of velocity.

The Excitation Force

It is a source of continuous supply of energy to the vibratory system. It is an external periodic force which acts on the vibratory system.

It is important to study the single degree freedom system for a clear understanding of basic features of a vibration problem.

Undamped Free Vibration

There are several methods to analyse an undapmed system.

Methodology

Method Based on Newton's II Law

According to the Newton's II law, the rate of change of linear momentum is proportional to the force impressed upon it

$$\frac{d}{dt}(mv) \;\alpha\; \text{Net force in direction of the velocity}$$

Using $$v = \frac{dx}{dt} = \dot{x}$$

$$\therefore \qquad \frac{d\dot{x}}{dt} = (m\ddot{x}) = c \sum F$$

where c is constant of proportionality.

or $$m\ddot{x} = c \sum F$$

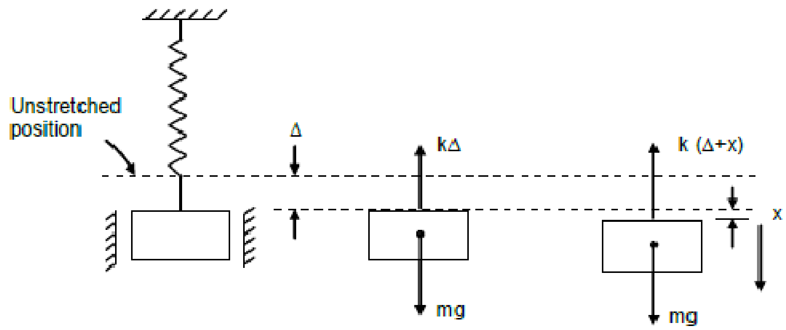
For proper units in a system $c = 1$

$$m\ddot{x} = \sum F$$

The direction of forces $m\ddot{x}$ and $\sum F$ are same. A model which represents undamped single degree of freedom system shall have two elements, i.e. helical spring and mass. The mass is constrained to move only in one direction as shown in Figure 7.2. The mass is in static condition in Figure 7.2(a). The free body diagram of the mass is shown in

Figure 7.2(b). The body is in equilibrium under the action of the two forces. Here 'Δ' is the extension of the spring after suspension of the mass on the spring.

Therefore, $$k\,\Delta = mg \qquad \ldots (7.1)$$



(a) Spring Mass (b) Static Condition (c) Dynamic Condition

Figure 7.2 : Undamped Free Vibration

Figure 7.2(c) represents the dynamic condition of the body. In this case, the body is moving down with acceleration '$\ddot{x}$' also in downward direction, therefore,

$$m\ddot{x} = \sum F \text{ in direction of } \ddot{x}$$

or $$m\ddot{x} = mg - k\,(x + \Delta) \qquad \ldots (7.2)$$

Incorporating Eq. (7.1) in Eq. (7.2)

$$m\ddot{x} = -\,k\,x$$

or $$m\ddot{x} + k\,x = 0 \qquad \ldots (7.3)$$

Method Based on D'Alembert's Principle

The free body diagram of the mass in dynamic condition can be drawn as follows :

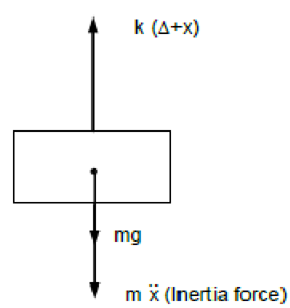


Figure 7.3 : Free Body Diagram

The free body diagram of mass is shown in Figure 7.3. The force equation can be written as follows :

$$m\ddot{x} + mg = k\,(x + \Delta) \qquad \ldots (7.4)$$

Incorporating Eq. (7.1) in Eq. (7.4), the following relation is obtained.

$$m\ddot{x} + kx = 0$$

This equation is same as we got earlier.

Energy Method

This method is applicable to only the conservative systems. In conservative systems there is no loss of energy and therefore total energy remains system is partly kinetic and partly potential (elastic strain energy). The kinetic energy is due to the mass (m) and velocity ($\dot{x}$). The potential e is due to spring stiffness and relative movement between the two ends spring.

$$\text{Energy } (E) = T + U = \text{constant } (C)$$

where T = Kinetic energy of the system, and'

U = Elastic strain energy.

Since total energy remains constant

$$\therefore \qquad \frac{dE}{dt} = 0 \quad \text{or} \quad \frac{d}{dt}(T + U) = 0$$

$$T = \frac{1}{2} m (\dot{x})^2$$

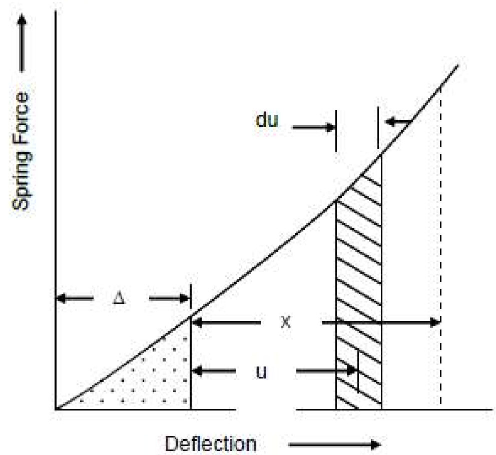


Figure 7.4 : Spring Force – Deflection Diagram

The potential energy of the system consists of two points :

(a) loss/gain in PE of mass, and

(b) strain energy of spring.

Consider an infinitesimal element du at $x = u$.

From Figure 7.4

$$\text{Spring force } (F_u) = k(u + \Delta)$$

$$\text{Work done } dW = k(u + \Delta) \times du$$

$$\therefore \qquad U = \int_0^x dW - \text{loss of PE of mass}$$

$$= \int_0^x k(u + \Delta)\, du - mg\, x$$

$$\therefore \qquad U = \int_0^x (ku + mg)\, du - mg\, x \qquad [\because k\Delta = mg]$$

$$\text{or} \qquad U = \frac{1}{2}(kx^2) + mg\, x - mg\, x$$

$$U = \frac{1}{2} kx^2$$

$$\frac{d}{dt}\left(\frac{1}{2}\, m\dot{x} + \frac{1}{2}\, kx^2\right) = 0$$

$\therefore$ $$\frac{1}{2}\, m \times 2\,\dot{x} \times \ddot{x} + \frac{1}{2}\, k \times 2x + \dot{x} = 0$$

or $$m\ddot{x} + kx = 0$$

This is the same equation as we got earlier.

Rayleigh's Method

It is a modified energy method. It may be noted that in a conservative system potential energy is maximum when kinetic energy is minimum and vice-versa. Therefore, equating maximum kinetic energy with maximum potential energy.

$\therefore$ $$\frac{1}{2}\, m\, (\dot{x}_{\max})^2 = \frac{1}{2}\, k\, (x_{\max})^2$$

and $$x_{\max} = X$$

$\therefore$ $$\frac{1}{2}\, m\, (X\,\omega)^2 = \frac{1}{2}\, k\, X^2$$

or $$\omega = \sqrt{\frac{k}{m}} \qquad \ldots (7.6)$$

Solution of Differential Equation

The differential equation of single degree freedom undamped system is given by

$$m\ddot{x} + kx = 0$$

or $$\ddot{x} + \left(\frac{k}{m}\right) x = 0 \qquad \ldots (7.7)$$

when coefficient of acceleration term is unity, the underroot of coefficient of x is equal to the natural circular frequency, i.e. 'ω_n'

$\therefore$ $$\omega_n = \sqrt{\frac{k}{m}} \qquad \ldots (7.8)$$

Therefore, Eq. (7.7) becomes

$$\ddot{x} + \omega_n^2\, x = 0 \qquad \ldots (7.9)$$

The equation is satisfied by functions $\sin \omega_n t$ and $\cos \omega_n t$. Therefore, solution of Eq. (7.9) can be written as

$$x = A \sin \omega_n t + B \cos \omega_n t \qquad \ldots (7.10)$$

where A and B are constants. These constants can be determined from initial conditions. The system shown in Figure 7.2(a) can be disturbed in two ways :

(a) by pulling mass by distance 'X', and

(b) by hitting mass by means of a fast moving object with a velocity \ say 'V'.

Considering case (a)

$$t = 0,\ x = X \ \text{ and } \ \dot{x} = 0$$

$\therefore$ $$X = B \ \text{ and } \ A = 0$$

$\therefore$ $$x = X \cos \omega_n t \qquad \ldots (7.11)$$

Considering case (b)

$$t = 0, \; x = 0 \; \text{and} \; \dot{x} = V$$

$$B = 0 \quad \text{and} \quad A = \frac{V}{\omega_n}$$

$$\therefore \qquad x = \frac{V}{\omega_n} \sin \omega_n t \qquad \ldots (7.12)$$

Behaviour of Undamped System

Consider the system shown in Figure 7.2(a). The system has been disturbed by pulling the mass by distance 'X'. The solution of the system in this case is given by Eq. (7.11) which is

$$x = X \cos \omega_n t$$

$$\therefore \qquad \dot{x} = -X\omega_n \sin \omega_n t = X\omega_n \cos\left(\omega_n t + \frac{\pi}{2}\right)$$

and $$\ddot{x} = -X\omega_n^2 \cos \omega_n t = X\omega_n^2 \cos(\omega_n t + \pi)$$

These expressions indicate that velocity vector leads displacement by $\frac{\pi}{2}$ and acceleration leads displacement by 'π'. The maximum velocity is $(X\omega_n)$ and maximum acceleration is $(X\omega_n^2)$.

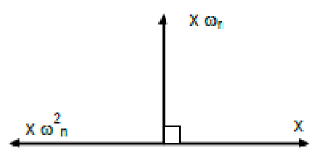


Figure 7.5

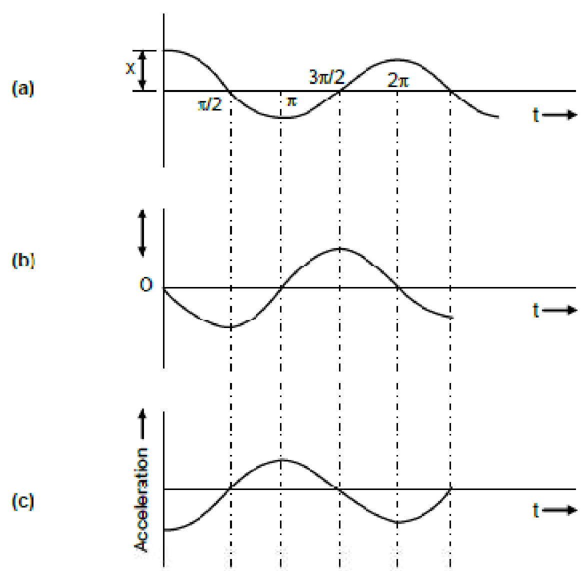


Figure 7.6 : Plots of Displacement, Velocity and Acceleration

Figure 7.6 shows the plots of displacement, velocity and acceleration, with respect to time. The following observations can be made from these diagrams :

(a) A body, if disturbed, will never stop vibrating.

(b) When displacement is maximum, velocity is zero and acceleration is maximum in direction opposite to displacement.

(c) When displacement is zero, velocity is maximum and acceleration is zero.

Damped Free Vibration

In undamped free vibrations, two elements (spring and mass) were used but in damped third element which is damper in addition to these are used. The three element model is shown in Figure 7.7. In static equilibrium

$$k\Delta = mg$$

$$m\ddot{x} = mg - k\,(x + \Delta) - c\dot{x}$$

$\therefore$ $$m\ddot{x} = -\,kx - c\dot{x}$$

or $$m\ddot{x} + c\dot{x} + kx = 0 \qquad \ldots (7.13)$$

Let $$x = X\,e^{st}$$

Substituting for x in Eq. (7.13) and simplifying it

$$ms^2 + cs + k = 0$$

or $$s^2 + \frac{c}{m}\,s + \frac{k}{m} = 0 \qquad \ldots (7.14)$$

$\therefore$ $$s_{1,2} = -\left(\frac{c}{2m}\right) \pm \frac{1}{2}\sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)} \qquad \ldots (7.15)$$

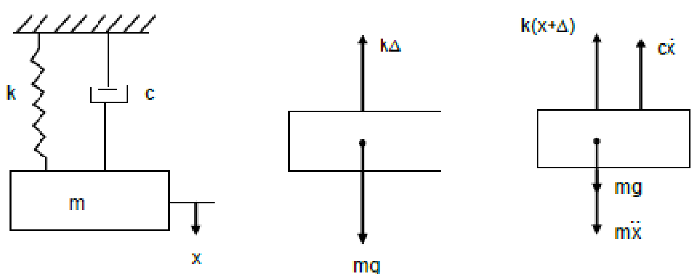


Figure 7.7 : Damped Free Vibration

The solution of Eq. (7.13) is given by

$$x = X_1\,e^{\left[-\left(\frac{c}{2m}\right) + \frac{1}{2}\sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)}\right]t} + X_2\,e^{\left[-\left(\frac{c}{2m}\right) - \frac{1}{2}\sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)}\right]t}$$

$$= e^{-\left(\frac{c}{2m}\right)t}\left[X_1\,e^{\frac{1}{2}\left\{\sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)}\right\}t} + X_2\,e^{-\frac{1}{2}\left\{\sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)}\right\}t}\right] \qquad \ldots (7.16)$$

The nature of this solution depends on the term in the square root. There are three possible cases :

(a) $\left(\frac{c}{m}\right)^2 > 4\left(\frac{k}{m}\right)$ – Overdamped case

(b) $\left(\frac{c}{m}\right)^2 = 4\left(\frac{k}{m}\right)$ – Critically damped case

(c) $\left(\frac{c}{m}\right)^2 < 4\left(\frac{k}{m}\right)$ – Underdamped case

Let the critical damping coefficient be C_c, therefore,

$$\left(\frac{C_c}{m}\right)^2 = 4\left(\frac{k}{m}\right)$$

or $$C_c = 2\sqrt{km} = 2\sqrt{\frac{k}{m}\,m^2} = 2m\sqrt{m^2}\sqrt{\frac{k}{m}} = 2m\,\omega_n$$

or $$C_c = 2\sqrt{km} = 2m\,\omega_n$$

Almost all the systems are underdamped in practice.

Therefore, $$\sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)} = i\sqrt{4\left(\frac{k}{m}\right) - \left(\frac{c}{m}\right)^2}$$

The ratio of damping coefficient (c) to the critical damping coefficient is called damping factor 'ζ'.

$$\zeta = \frac{C}{C_c} \qquad \ldots (7.17)$$

$$\sqrt{4\omega_n^2 - \left(\frac{c}{C_c} \times \frac{C_c}{m}\right)^2} = 4\sqrt{4\omega_n^2 - \zeta^2 \times \left(\frac{2m\,\omega_n}{m}\right)^2}$$

$$= 2\omega_n\sqrt{1-\zeta^2}$$

$\therefore$ $$x = e^{-\frac{c}{2m}t}\left[X_1\,e^{(i\,\omega_n\sqrt{1-\zeta^2})\,t} + X_2\,e^{(-i\,\omega_n\sqrt{1-\zeta^2})\,t}\right]$$

Let $$\omega_n\sqrt{1-\zeta^2} = \omega_d \quad \text{(say)} \qquad \ldots (7.18)$$

where ω_d is natural frequency of the damped free vibrations.

Therefore, for under-damped case

$$x = e^{-\frac{c}{2m}}\left[X_1\,e^{i\,\omega_d\,t} + X_2\,e^{-i\,\omega_d\,t}\right] \qquad \ldots (7.19)$$

For critically damped system

$$x = (X_1 + X_2\,t)\,e^{-\frac{c}{2m}t} \qquad \ldots (7.20)$$

For overdamped system

$$x = e^{-\frac{c}{2m}t}\left[X_1\,e^{\omega_n\{\sqrt{\zeta^2-1}\}\,t} + X_2\,e^{-\omega_n\{\sqrt{\zeta^2-1}\}\,t}\right] \qquad \ldots (7.21)$$

$$\frac{C}{2m} = \frac{C}{C_c} \times \frac{C_c}{2m} = \zeta \times \frac{2m\,\omega_n}{2m} = \zeta\,\omega_n$$

$\therefore$ $$x = e^{-\zeta\,\omega_n\,t}\left[X_1\,e^{\omega_n\,t\sqrt{\zeta^2-1}} + X_2\,e^{-\omega_n\,t\sqrt{\zeta^2-1}}\right] \qquad \ldots (7.22)$$

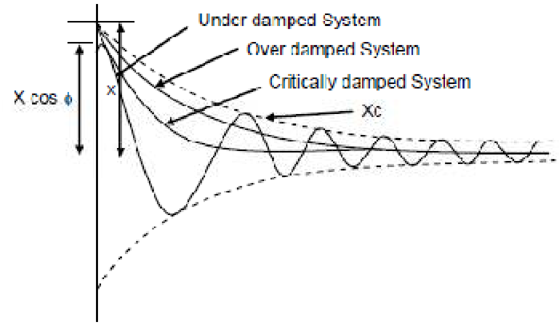


Figure 7.8

The Eq. (7.19) can also be written as

$$x = X e^{-\zeta \omega_n t} \cos(\omega_d t + \phi) \qquad \ldots (7.23)$$

where X and ϕ are constants. X represents amplitude and ϕ phase angle.

Let at $\quad t = t, \; x = x_0.$

$$\therefore \quad x_0 = X e^{-\zeta \omega_n t} \cos(\omega_d t + \phi) \qquad \ldots (7.24)$$

After one time period

$$t = t + t_p \quad \text{and} \quad x = x_1$$

$$\therefore \quad x_1 = X e^{-\zeta \omega_n (t + t_p)} \cos\{\omega_d (t + t_p) + \phi\} \qquad \ldots (7.25)$$

Dividing Eq. (7.24) by Eq. (7.25)

$$\frac{x_0}{x_1} = \frac{X e^{-\zeta \omega_n (t + t_p)} \cos \omega_d t + \phi}{X e^{-\zeta \omega_n (t + t_p)} \cos\{\omega_d (t + t_p) \phi\}}$$

Since $\quad t_p = \dfrac{1}{f_p} = \dfrac{2\pi}{\omega_d}$

or $\quad \omega_d t_p = 2\pi$

$$\therefore \quad \frac{x_0}{x_1} = e^{\zeta \omega_n t_p} \frac{\cos(\omega_d t + \phi)}{\cos\{\omega_d t + 2\pi + \phi\}}$$

Since $\quad \cos\theta = \cos(2\pi + \theta)$

$$\therefore \quad \cos(\omega_d t + \phi) = \cos\{\omega_d t + 2\pi + \phi\}$$

$$\therefore \quad \frac{x_0}{x_1} = e^{\zeta \omega_n t_p}$$

or $\quad L_n\left(\dfrac{x_0}{x_1}\right) = \zeta \omega_n t_p = \zeta \omega_n \dfrac{2\pi}{\omega_d} = \dfrac{2\pi \omega_n \zeta}{\omega_n \sqrt{1 - \zeta^2}}$

or $$L_n\left(\frac{x_0}{x_1}\right) = \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}} \qquad \ldots (7.26)$$

$\dfrac{2\pi \zeta}{\sqrt{1 - \zeta^2}}$ is called logarithmic decrement.

If at $\quad t = t + n t_p$

It can be proved that

$$L_n \frac{x_0}{x_n} = \frac{2n\,\pi\,\zeta}{\sqrt{1-\zeta^2}} \qquad \ldots (7.27)$$

If $\qquad \zeta < 0.3 \quad L_n \dfrac{x_0}{x_1} \approx 2\pi\,\zeta$

Figure 7.8 represents displacement time diagram for the above mentioned three cases. For over-damped and critically damped system mass returns to its original position slowly and there is no vibration. Vibration is possible only in the under-damped system because the roots of Eq. (7.14) are complex and solution consists of periodic functions (Eq. (7.22)).

MECHANICAL VIBRATIONS

UNIT – III

Forced Vibrations of Single Degree of Freedom Systems

Sources of Excitation , Equations of motion , Response of undamped system under harmonic excitation, Total response , beating phenomenon , Response of damped system under harmonic excitation, frequency response , quality factor and band width , response under harmonic excitation of the base, vibration isolation, transmissibility, force transmission to foundations, response of a damped system under rotating unbalance.

FORCED VIBRATION DUE TO HARMONIC EXCITATION

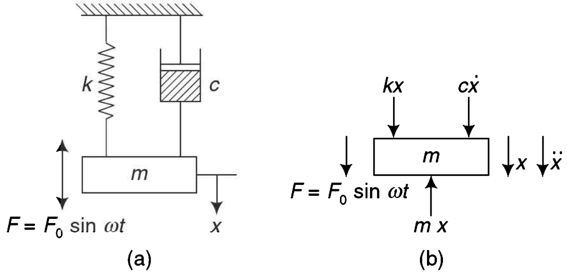


Fig. 4.1 *Spring-mass-dashpot system under forced vibration*

Let us consider a classical spring-mass-dashpot system excited by a sinusoidal forcing function $F = F_0 \sin \omega t$ as shown in Fig. 4.1(a) where 'F'$_0$ is the amplitude and 'ω' is the angular frequency. Let 'k' be the spring stiffness of the spring, 'm' be the mass of the body and 'c' be the damping coefficient. Let at any instant the system be displaced through a distance 'x' from the equilibrium position as shown in Fig. 4.1(a). The body has at the instant a velocity '$\dot{x}$' at the instant in the upward direction, i.e. the direction of positive of 'x' where the external force ($F = F_0 \sin \omega t$) is acting on the system. The forces acting are as shown in the free-body diagram Fig. 4.1(b).

The damping resistance at any instant is equal to $c\dot{x}$.

Note: In case of forced vibrations, there will be four forces acting on a system, i.e. spring force, damping force, inertia force and impressed force or external force. (See Eq. 4.4).

Now applying Newton's second law of motion to mass 'm', i.e. $\Sigma F = m\ddot{x}$

$$-kx - c\dot{x} + F = m\ddot{x}, \quad m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t \qquad \text{...4.1}$$

This is a linear differential equation of motion, which is a second-order nonhomogeneous differential equation of a single-degree freedom system having free vibration with damping. The complete solution of this equation has two components, viz. complimentary function x_c (CF) and particular integral or function x_p (PI),

i.e. $\quad x = x_c + x_p$

1. *Complimentary function 'x_c'* (See Sec. 3.6, Chapter 3)

This can be obtained by equating the left-hand side of Eq. 4.1 to zero. That means there is no forcing function ($F = F_0 \sin \omega t$) on the system. This is also called *transient response* because it will eventually die out.

The resulting equation is $m\ddot{x} + c\dot{x} + kx = 0$.

This equation is a linear, fundamental homogeneous second-order differential equation of motion of a single-degree-of-freedom system having free vibration with damping.

$$\therefore \quad \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0. \quad \text{Here } \frac{c}{m} = \frac{c}{c_c}\cdot\frac{c_c}{m} = \xi\cdot\frac{2m\omega_n}{m} = 2\xi\omega_n,\ \frac{k}{m} = \omega_n^2$$

where ξ = Damping ratio $\xi = \frac{c}{c_c}\cdot$ Using these values in above equation, we have

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0 \qquad \text{...4.2}$$

Let $\quad x = Ae^{st},\ \dot{x} = Ase^{st} = sx,\ \ddot{x} = A\,s^2e^{st} = s^2x$

Using these values in Eq. 4.2,

$$s^2x + 2\xi\omega_n s\,x + \omega_n x = 0,\ (s^2 + 2\,\xi\omega_n s + \omega_n^2)x = 0, \text{ as } x \neq 0$$

$$\therefore \quad s^2x + 2\xi\omega_n + \omega_n^2 = 0$$

This is a quadratic equation of 's' and there will be two roots for 's'

$$\therefore \quad s_{1,\,2} = \frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2 - 4\omega_n^2}}{2}$$

$$\therefore \quad s_{1,\,2} = -\xi\omega_n \pm \sqrt{(\xi\omega_n)^2 - \omega_n^2}$$

$$\therefore \quad s_{1,\,2} = \omega_n\left(-\xi \pm \sqrt{\xi^2 - 1}\right) \qquad \text{...4.3}$$

When $\xi < 1$, (underdamping) in Eq. 4.3, the solution is

$$x = A_1e^{s_1t} + A_2e^{s_2t} \text{ or } x = e^{-\xi\omega_nt}(c_1\cos\omega_dt + c_2\sin\omega_dt)$$

$$s_{1,2} = -\xi\omega_n \pm i\,\omega_d \text{ where } \omega_d = \omega_n\sqrt{1-\xi^2} \text{ or } x = Ye^{-\xi\omega_nt}\sin(\omega_dt + \psi)$$

where $\quad Y = \sqrt{c_1^2 + c_2^2} \quad \psi = \tan^{-1}\left(\frac{c_2}{c_1}\right)$

when $\xi = 1$ (critical damping), the solution is $x = (c_1 + c_2t)e^{-\omega_nt}$,

When $\xi > 1$, (overdamping), the solution is

$$x = A_1e^{s_1t} + A_2e^{s_2t},\ s_{1,\,2} = \omega_n(-\xi \pm \sqrt{\xi^2 - 1}).$$

2. Particular integral or solution 'x_p' (steady–state component) Let $x = X\sin(\omega t\text{-}\phi)$, be the particular solution (because the forcing function is a sinusoidal, the particular integral should also be sinusoidal) where 'X' is the constant amplitude of vibration of the system and 'ϕ' is the angle (phase difference) by which the displacement vector lags the force vector and 'ω' is the angular frequency.

$x = X\sin(\omega t - \phi)$. Differentiating with respect to time 't' twice,

$$\dot{x} = \omega X\cos(\omega t - \phi) = \omega X\sin\left(\omega t - \phi + \frac{\pi}{2}\right)$$

$$\ddot{x} = -\omega^2X\sin(\omega t - \phi)$$

Substituting these values in Eq. 4.1, we get

$$m\,[-\omega^2X\sin(\omega t - \phi)] + c\left[\omega X\sin\left(\omega t - \phi + \frac{\pi}{2}\right)\right] + kX\sin(\omega t - \phi) = F_0\sin\omega t$$

$$mX\omega^2\sin(\omega t - \phi + \pi)] + cX\omega\sin\left(\omega t - \phi + \frac{\pi}{2}\right) + kX\sin(\omega t - \phi) - F_0\sin\omega t = 0 \qquad \text{...4.4}$$

Inertia force + Damping force + Spring force – Impressed force = 0

From the above equation we absorbed that

- The term $mX\,\omega^2 \sin(\omega t - \phi + \pi)$ is the inertia force
- The term $cX\,\omega \sin\left(\omega t - \phi + \frac{\pi}{2}\right)$ is the damping force
- The term $kX \sin(\omega t - \phi)$ is the spring force
- The term $F_0 \sin \omega t$ is harmonic excitation force (impressed Force)

These forces can be vectorially represented as follows:

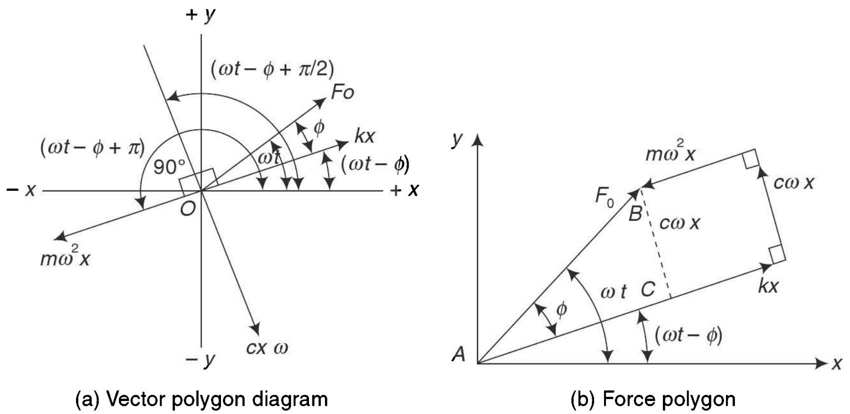


Fig. 4.2 *Vector representation of forces on the system having forced vibration*

From vector diagram Fig. 4.2(a), we can observe that

(a) Spring force is always opposite to the displacement

(b) Damping force lags the displacement by 90°

(c) Inertia force is out of phase with the displacement (180°)

In Fig. 4.2(b), force polygon from the right–angled triangle ABC

$$AB^2 = BC^2 + AC^2, \text{ where } AB = F_0, \quad BC = c\,\omega X, AC = (kX - m\omega^2 X)$$

$$F_0^2 = (c\,X\,\omega)^2 + (kX - mX\omega^2)^2, \quad F_0^2 = X^2[(k - m\omega^2)^2 + (c\omega)^2]$$

$$X^2 = \frac{F_0^2}{(k - m\omega^2)^2 + (c\omega)^2}$$

The steady state amplitude $X = \sqrt{\dfrac{F_0^2}{(k - m\omega^2)^2 + (c\omega)^2}}$

$$X = \frac{F_0}{\sqrt{k^2\left(1 - \frac{m}{k}\,\omega^2\right)^2 + k^2\left(\frac{c}{k}\omega\right)^2}} = \frac{F_0}{k\sqrt{\left(1 - \frac{m}{k}\,\omega^2\right)^2 + \left(\frac{c}{k}\,\omega\right)^2}}$$

Dividing the right–hand side numerator and denominator by 'k', we get

$$X = \frac{\frac{F_0}{k}}{\sqrt{\left(1 - \frac{m}{k}\omega^2\right)^2 + \left(\frac{c}{k}\omega\right)^2}} \quad \ldots 4.5$$

The static deflection 'X_{st}' due to the harmonic force is given by

$$X_{st} = \frac{F_0}{k}, \text{ and } \frac{m}{k} = \frac{1}{\omega_n^2}, \frac{c}{k} = \frac{c}{c_c} \times \frac{c_c}{k} = \xi \times \frac{2\sqrt{mk}}{k} = \xi \times \frac{2\sqrt{m}}{\sqrt{k}} = 2\xi\sqrt{\frac{m}{k}} = \frac{2\xi}{\omega_n}, \frac{c}{k} = \frac{2\xi}{\omega_n}$$

Using these values in Eq. 4.5,

$$\ddot{x} + 2\xi\omega_n \dot{x} + \omega_n^2 x = 0 \quad \ldots \text{(Eq. 4.2)}$$

$$X = \frac{X_{st}}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \quad \frac{X}{X_{st}} = \frac{1}{\sqrt{(1 - r^2) + (2\xi r)^2}}$$

Let $r = \frac{\omega}{\omega_n}$ where 'r' is the frequency ratio.

Once again from force diagram Fig. 4.2(b) in the right–angled triangle ABC,

$$\tan\phi = \frac{BC}{AC} = \frac{cX\omega}{kX - mX\omega^2}$$

$$\tan\phi = \frac{c\omega}{k - m\omega^2} = \frac{c\omega}{k\left(1 - \frac{m}{k\omega^2}\right)}$$

$$= \frac{\left[\frac{c}{k}\right]\omega}{1 - \frac{m}{k}\omega^2}, = \frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]}$$

$$\therefore \quad \tan\phi = \frac{2\xi r}{(1 - r^2)} \text{ or } \phi = \tan^{-1}\left[\frac{2\xi r}{1 - r^2}\right] \quad \ldots 4.6$$

Considering the underdamped case ($\xi < 1$), the complete solution is given by

$$x = x_c + x_p$$

$$x = X_1 e^{-\xi\omega_n t}\sin(\omega_d t + \psi) + X\sin(\omega t - \phi)$$

where $$X = \frac{X_{st}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}, \phi = \tan^{-1}\left[\frac{2\xi r}{1 - r^2}\right]$$

$$\omega_d = \omega_n\sqrt{1 - \xi^2} \quad \text{where } X_{st} = \frac{F_0}{k}.$$

MAGNIFICATION FACTOR (MF)

Magnification factor (MF) is defined as the ratio of steady–state amplitude to the zero frequency deflection (static deflection due to harmonic force),

i.e. Magnification factor, $\text{MF} = \frac{X}{X_{\text{st}}}, \frac{X}{X_{\text{st}}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$ ... 4.7

The phase lag, $\phi = \tan^{-1}\left(\frac{2\xi r}{1-r}\right)$ ... 4.8

where $r = \frac{\omega}{\omega_n}$

Dimensionless plots of magnification factor (MF) versus frequency ratio 'r' and phase lag 'ϕ' versus frequency ratio 'r' for different values of damping factor 'ξ'.

Magnification factor and phase lag 'ϕ' are the functions of 'ξ' and frequency ratio $\frac{\omega}{\omega_n} = r$.

Case (i) When $r = 0$, (at zero frequency) in Eq. 4.7, Eq. 4.8 becomes

$$\frac{X}{X_{st}} = \frac{1}{(\sqrt{1-0})^2 + 0} = 1 \text{ or } X = X_{st}$$

which is the definition of zero frequency deflection and

$$\phi = \tan^{-1}(0),\ \phi = 0.$$

$\therefore$ the amplitude ratio or magnification factor is independent of the damping ratio 'ξ'.

Case (ii) When $r = 1$ (resonance, i.e. $\omega = \omega_n$) in Eq. 4.7, Eq. 4.8 becomes

$$\frac{X}{X_{st}} = \frac{1}{\sqrt{0 + (2\xi)^2}} = \frac{1}{2\xi} \text{ and } \phi = \tan^{-1}\left(\frac{2\xi r}{-r}\right) = \tan^{-1}(\infty) = 90°.$$

The amplitude ratio or magnification factor depends on damping ratio 'ξ' at resonance. As the value of 'ξ' decreases, magnification factor increases and the converse is true, but at $\xi = 0$, i.e. for undamped system, the value of magnification factor $= \infty$.

Case (iii) When $r >> 1$, and $r^2 >>> 1$

$$\therefore \qquad \frac{1}{r^2} <<< 1 \text{ or } \frac{1}{r^2} \approx 0 \text{ and } \frac{1}{r} \approx 0$$

$\therefore$ equations 4.7 and 4.8 become

$$\therefore \qquad \frac{X}{X_{st}} = \frac{1}{\sqrt{r^2\left(\frac{1}{r^2} - 1\right) + (2\xi)^2}} \text{ and } \phi = \tan^1\left[\frac{2\xi r}{r^2\left(\frac{1}{r^2} - 1\right)}\right]$$

$$\therefore \qquad \frac{X}{X_{st}} = \frac{1}{r\sqrt{\left(\frac{1}{r^2} - 1\right) + (2\xi)^2}} \text{ and } \phi = \tan^{-1}\left[\frac{2\xi r}{-r^2}\right],$$

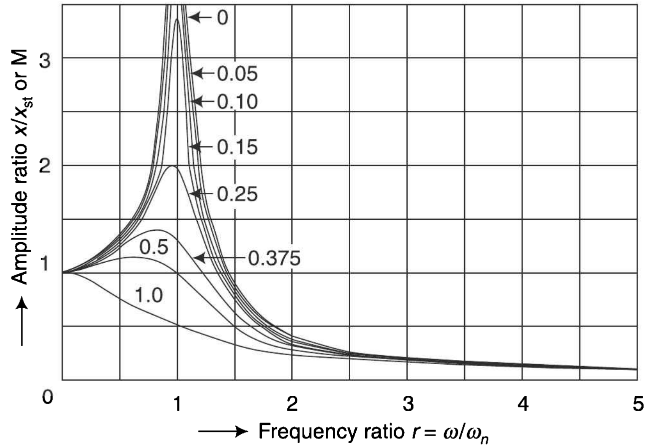
$$\text{MF} = \frac{X}{X_{st}} \approx 0 \text{ and } \phi = \tan^{-1}[-0] = 180^\circ$$

The dimensionless plots of **magnification factor** (MF) versus **frequency ratio(r)** and **phase lag (ϕ)** versus **frequency ratio(r)** for different values of damping factor are shown in Fig. 4.3(a). These curves reveal a lot of interesting and useful information regarding the behaviour of the system to sinusoidal excitation.

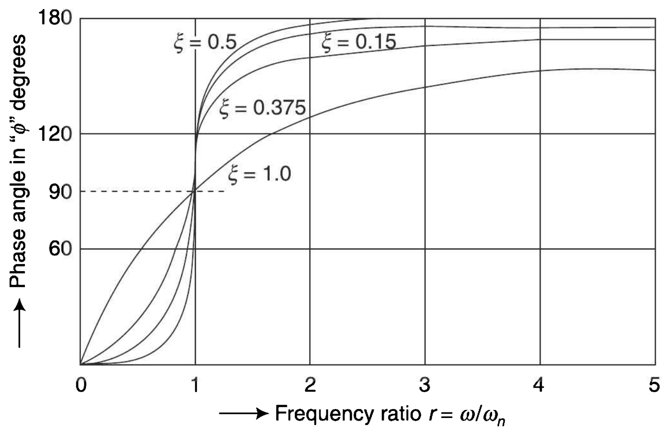
Curves of Fig. 4.3(a) are also known as *frequency-response curves,* since they give the response of the system to various frequencies. It is seen from these curves that the response of a particular system at any particular frequency is lower for higher value of damping and lies below those for lower values of damping. At zero frequency the magnification factor is unity and is independent of the damping, i.e. $X = X_{st}$, which itself is the definition of zero frequency deflection. At very high frequency, the magnification factor tends to zero or the amplitude of vibration becomes very small. At resonance ($\omega = \omega_n$), the amplitude of vibration becomes excessive for small damping and decreases with increase in damping. For zero damping at resonance, the amplitude is infinite theoretically. Practically, however the system may go into destruction much before that or the amplitude may be limited because of other factors.

The phase angle also varies from zero at low frequencies to 180° at very high frequencies. It is 90° at resonance and is independent of damping. Over a small frequency range containing the resonance point, the variation of phase angle is more abrupt for lower values of damping than for higher values. The more abrupt the change in phase angle about resonance, sharper is the peak in the frequency response curve. For zero damping, the phase lag suddenly changes from zero to 180° at resonance. The corresponding zero damping frequency-response curve is also infinitely sharp at resonance.

Let us now study the phenomenon of Fig. 4.3(b) by means of the vector diagram and gain some more insight into what is happening in the system. With reference vector diagram at very low frequency (ω is very small), the inertia term '$m\omega^2 x$' becomes



(a) Magnification factor versus frequency ratio for different amount of damping.



(b) Frequency ratio versus phase angle

Fig. 4.3

negligibly small and damping term '$c\omega X$' is also small. This gives rise to small value of 'ϕ' as shown in Fig. 4.3(b).

The impressed force 'F_0' is almost equal and opposite to the spring force 'kx' under these conditions. Thus, for very low frequencies, the phase angle tends to zero and the impressed force wholly balances the spring force. With increase in the frequency, the damping force vector ($c\omega X$) grows larger. Angle 'ϕ' has also to increase so that component of 'F_0' perpendicular to x-direction may balance the increasing damping force. The inertia force vector grows much more rapidly with increase in frequency because of the factor 'ω^2' in its expression. If we continue to increase the frequency, a time comes when the spring force and inertia force vectors are equal and opposite as shown in the figure and this condition is $kx = m\omega^2 x$ or $\omega = \sqrt{\frac{k}{m}} = \omega_n$. This is the resonance condition of the system and the vector diagram becomes rectangular. The impressed force completely balances the damping force and $\phi = 90°$.

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}},\ (1.435)^2 = \frac{4\pi\xi^2}{1-\xi^2},\ \xi = 0.2236,\quad \omega_n = 3.58 \text{ rad/s},\ r = \frac{\omega}{\omega_n} = \frac{3}{3.58} = 0.84$$

The amplitude of vibration is given by $\frac{X}{X_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$

$$\frac{X}{X_{st}} = \frac{1}{\sqrt{(1-3.58^2)^2 + (2 \times 2236 \times 3.58)^2}},\ X_{st} = F_0/k = 20/580$$

$\therefore X = 0.072$ m = 7.2 mm

Phase angle is given by $\phi = \tan^{-1}\left(\frac{2\xi r}{1-r^2}\right) = \tan^{-1}\left(\frac{2 \times 0.2236 \times 3.58}{1-3.58^2}\right),\ \phi = 51.53°$

ROTATING AND RECIPROCATING UNBALANCE

Unbalance in reciprocating machinery is a common source of vibration. Consider a machine of mass 'm' mounted on a foundation of stiffness 'k' and damping coefficient 'c' as shown in Fig. 4.4(a) The unbalance is represented by the reciprocating mass 'm' having cranked small rotation 'e' and connecting rod length 'l'. Let 'ω' be the angular velocity of the crank. Let 'x' be the displacement of non reciprocating mass 'M-m' at any instant of time 't'. The displacement of reciprocating mass 'm' from static equilibrium position is given by $x + e \sin \omega + \frac{e}{l} \sin 2\,\omega t + ...$

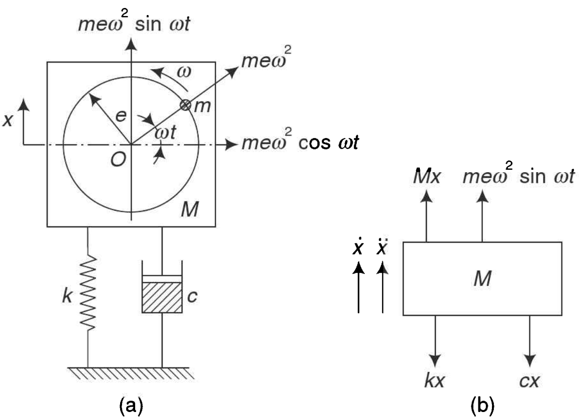


Fig. 4.4 *Rotating unbalance*

Applying D'Alembert's principle or Newton's second law of motion to mass 'M'

$$\Sigma F = M\ddot{x} \quad \therefore c\dot{x} + kx - me\omega^2 \sin \omega t = -M\ddot{x} \quad M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin \omega t \quad ...4.9$$

or let $\quad me\omega^2 = F_0$

$$\therefore \quad M\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t \quad ...4.10$$

This is a second-order nonhomogeneous differential equation of motion, whose solution is given by

$$x = x_c + x_p$$

where x_c = Complementary function (transient response)

x_p = Particular integral (steady-state response already discussed in Article 4.2, case 'ii')

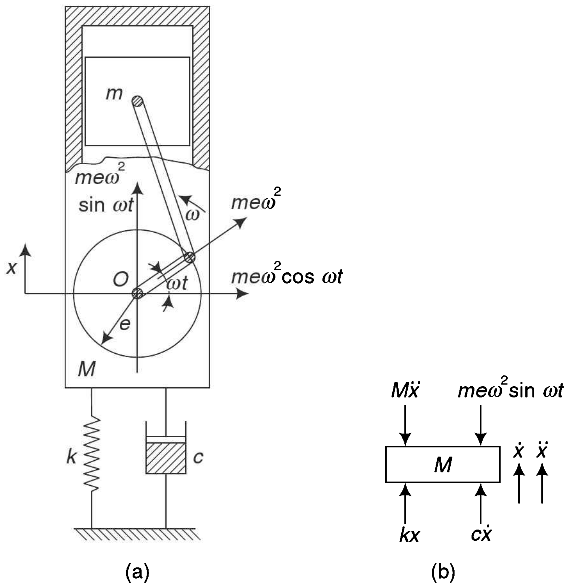


Fig. 4.5 *Reciprocating unbalance*

Considering the steady-state response or to find the particular integral 'x_p',

let $\quad x_p = x = X \sin(\omega t - \phi)$.

where 'X' is the amplitude of vibration of the system, 'ϕ' is the angle by which the displacement vector lags the force vector, and 'ω' is the angular frequency in rad/s.

$$\dot{x} = \omega X \cos(\omega t - \phi) = \omega X \sin\left(\omega t - \phi + \frac{\pi}{2}\right) \ddot{x} = -\omega^2 X \sin(\omega t - \phi).$$

Substituting these value in Eq. (4.9),

$$M[-\omega^2 X \sin(\omega t - \phi)] + c\left[\omega X \sin\left(\omega t - \phi + \frac{\pi}{2}\right)\right] + kX \sin(\omega t - \phi) = F_0 \sin \omega t$$

Rearranging the above term we have,

$$M\omega^2 X \sin(\omega t - \phi + \pi) + cX\omega \sin\left(\omega t - \phi + \frac{\pi}{2}\right) + kX \sin(\omega t - \phi) - F_0 \sin \omega t = 0$$

From the above equation, we absorbed that

Inertia force + Damping force + Spring force – Impressed force = 0

The term $mX\omega^2 \sin(\omega t - \phi + \pi)$ is the inertia force.

The term $cX\omega \sin(\omega t - \phi + \pi/2)$ is the damping force.

The term $kX \sin(\omega t - \phi)$ is spring force.

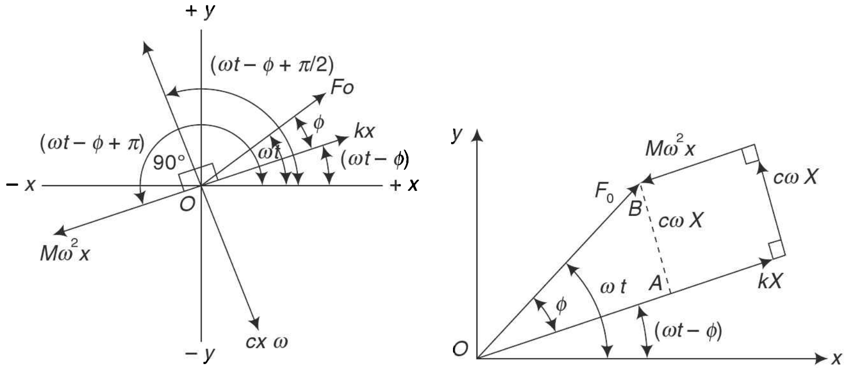
The term $F_0 \sin \omega t$ is harmonic excitation force (impressed force).

These forces can be vectorially represented as follows.

From the force polygon, $\overrightarrow{OB} = F_0$, $\overrightarrow{AB} = c\omega X$, $\overrightarrow{OA} = kX - M\omega^2 X$

In force polygon, from right-angled triangle OAB,

$$OB^2 = OA^2 + AB^2 \quad \therefore \; F_0^{\,2} = (kX - M\omega^2 X)^2 + (c\omega X)^2$$



(a) Vector polygon

(b) Force polygon

Fig. 4.6 *Force and vector polygon*

$$F_0^{\,2} = X^2\left[(k - M\omega^2)^2 + (c\omega)^2\right] X^2 = \frac{F_0^2}{(k - M\omega^2)^2 + (c\omega)^2},\; X = \frac{F_0}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}},$$

But $F_0 = me\omega^2$.

$$\therefore \qquad X = \frac{me\omega^2}{\sqrt{k^2\left[\left(1 - \frac{M}{k}\omega^2\right)^2 + \left(\frac{c}{k}\omega\right)^2\right]}},\; X = \frac{me\omega^2}{k\sqrt{\left(1 - \frac{M}{k}\omega^2\right)^2 + \left(\frac{c}{k}\omega\right)^2}}$$

Dividing the right-hand side by both numerator and denominator by 'k', we get

$$X = \frac{\dfrac{me\omega^2}{k}}{\sqrt{\left(1 - \frac{M}{k}\omega^2\right)^2 + \left(\frac{c}{k}\omega\right)^2}}$$

We have $\dfrac{me}{k}$

Dividing and multiplying by 'M', we have $\dfrac{m}{M}\cdot\dfrac{M}{k}\cdot e$

But $\dfrac{k}{M} = \omega_n^2$ or $\dfrac{M}{k} = \dfrac{1}{\omega_n^2}\cdot$ Divide and multiply by c_c

$$\therefore \; \frac{me}{k} = \frac{m}{M}\cdot\frac{e}{\omega_n^2}\;\frac{me}{k} = \frac{m}{M}\cdot\frac{e}{\omega_n^2}\times\frac{c_c}{c_c}\;\frac{M}{k} = \frac{1}{\omega_n^2} \text{ and } \frac{c}{k} = \frac{c}{c_c}\cdot\frac{2M\omega_n}{k} = \xi\cdot\frac{2\omega_n}{\omega_n^2} = \frac{2\xi}{\omega_n}$$

$$\therefore \; X = \frac{\dfrac{me}{M}\left(\dfrac{\omega}{\omega_n}\right)^2}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}\cdot \qquad \text{Let } r = \frac{\omega}{\omega_n} \quad \therefore \; \frac{MX}{me} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} \quad \text{...4.11}$$

And $$\tan\phi = \frac{AB}{OA} = \frac{cx\omega}{kx - Mx\omega^2} = \frac{c\omega}{k - M\omega^2} = \frac{\frac{c}{k\omega}}{1 - \frac{M}{k}\omega^2} = \frac{2\xi r}{1 - r^2}$$

$$\therefore \qquad \phi = \tan^{-1}\left[\frac{2\xi r}{1 - r^2}\right] \qquad \text{...4.12}$$

Discussion on $\frac{MX}{me}$ Versus $\frac{\omega}{\omega_n}$.

Case (i) When $r = 0$ in Eq. 4.11

$\frac{Mx}{me} = 0$, which is independent of the damping ratio 'ξ'.

Case (ii) When $r = 1$ (Resonance, i.e. $\omega = \omega_n$) in Eq. 4.11

$\frac{Mx}{me} = \frac{1}{\sqrt{0 + (2\xi)^2}}$, $\frac{Mx}{me} = \frac{1}{2\xi}$, which depends on the damping ratio 'ξ', at $\xi = 0$, $\frac{Mx}{me} = \infty$.

i.e. at resonance and at damping ratio $= 0$, $\frac{Mx}{me} = \infty$.

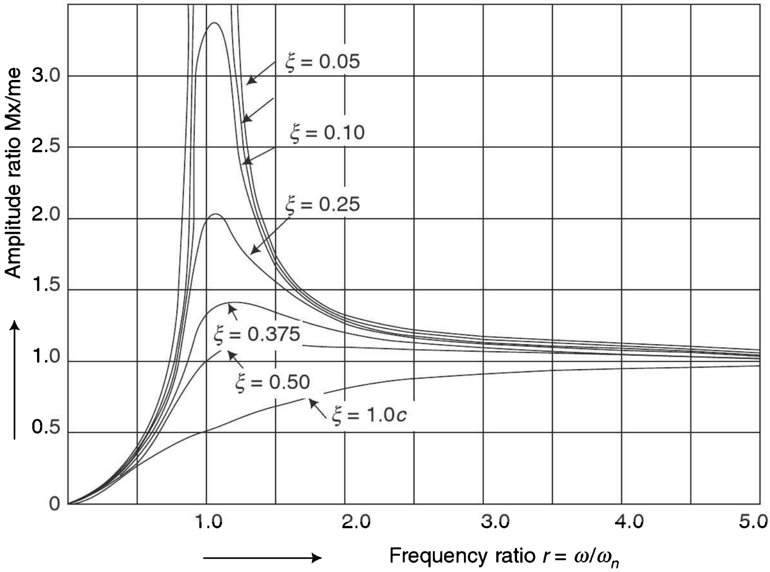


FIG. 4.7 *Amplitude ratio versus frequency ratio*

Case (iii) When $r >> 1$ ($\omega >> \omega_n$), i.e. $r^2 >>> 1$

$$\therefore \qquad \frac{1}{r^2} <<< 1, \quad \text{i.e. } \frac{1}{r^2} \approx 0 \text{ from Eq. 4.11}$$

$$\therefore \qquad \frac{Mx}{me} = \frac{r^2}{\sqrt{r^4\left(\frac{1}{r^2} - 1\right)^2 + r^4\left(\frac{2\xi}{r^2}\right)^2}}, \quad \frac{Mx}{me} = \frac{r^2}{r^2\sqrt{\left(\frac{1}{r^2} - 1\right)^2 + \left(\frac{2\xi}{r^2}\right)^2}} \quad \therefore \frac{Mx}{me} \approx 1$$

FORCED VIBRATION DUE TO EXCITATION OF THE SUPPORT MOTION

In most of locomotives and vehicles, the wheels are mounted on a base or support for the systems. These wheels can move vertically up and down on the surface of the base or support on the surface during the moving of the vehicle. In the motion of these vehicles or body of the wheels and base or support, there is a relative motion between the support motion, relative to the wheels and the wheels are having motion relative to the road surface.

In case of support motion, the amplitude of the motion depends upon the speed of the vehicle and the nature of the road surface. The vibration measuring instruments are designed on the support motion approach. In a vibratory system where the support is put to excitation, (i) absolute motion, and (ii) relative motion become most important. Such systems are supported to have a spring-mass-damper system of a single degree of freedom with a moving support or base as shown in Fig. 4.8.

1. Absolute motion (motion transmissibility) Absolute motion of a mass means its motion with respect to the coordinate system attached to the earth as shown in Fig. 4.8(a). The absolute displacement of support is $y = Y \sin \omega t$[sinusoidal motion Fig. 4.8(b)] and the absolute displacement of the mass 'm' from its equilibrium position is 'x'. The displacement of the mass 'm' relative to the support is 'z'.

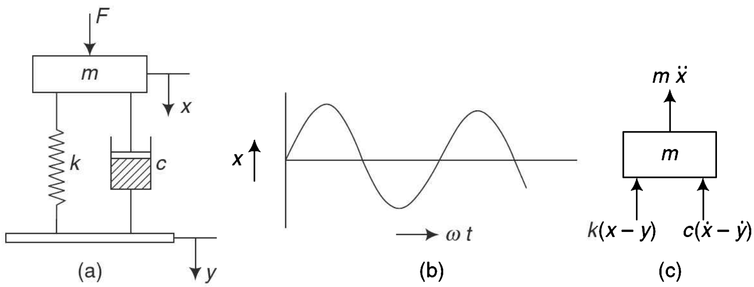


Fig. 4.8 *Support motion*

The net elongation of the spring is $(x - y)$ and the relative motion between the two ends of the damper is $(\dot{x} - \dot{y})$. Then $z = (x - y)$ and $\dot{z} = (\dot{x} - \dot{y})$.

Let us consider a spring-mass-damper system subjected to the support motion as shown in Fig. 4.8(a). Due to the support motion, let 'x' be the absolute motion of the system at any instant of time and respective FBD is as shown in Fig. 4.8(c).

Now apply Newton's second law of motion to mass 'm' $\Sigma F = m\ddot{x}$

$$m\ddot{x} = -c(\dot{x} - \dot{y}) - k(x - y), \; m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \qquad \text{... 4.13}$$

Since $\qquad y = Y \sin \omega t, \dot{y} = \omega Y \cos \omega t$

Substituting these values in Eq. 4.13, we get

$$m\ddot{x} + c\dot{x} + kx = c\omega Y \cos \omega t + kY \sin \omega t, \;\; m\ddot{x} + c\dot{x} + kx = Y(c\omega \cos \omega t + k \sin \omega t)$$

Multiplying and dividing by right-hand side by $\sqrt{k^2 + (c\omega)^2}$ and simplifying, we get

$$= Y\sqrt{k^2 + (c\omega)^2}\left[\frac{k}{\sqrt{k^2 + (c\omega)^2}} \sin \omega t + \frac{c\omega}{\sqrt{k^2 + (c\omega)^2}} \cos \omega t\right]$$

$$= Y\sqrt{k^2 + (c\omega)^2}\,\{\sin \omega t \cos \alpha + \cos \omega t \sin \alpha\}$$

$$= Y\sqrt{k^2 + (c\omega)^2}\,\sin(\omega t + \alpha)$$

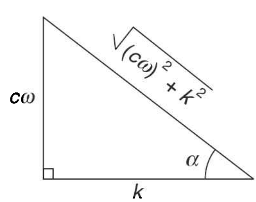


Fig. 4.9 *Absolute motion*

$$\therefore \qquad m\ddot{x} + c\dot{x} + kx = Y\sqrt{k^2 + (c\omega)^2}\,\sin(\omega t + \alpha) \qquad \text{... 4.14}$$

and phase angle is given by $\tan \alpha = \dfrac{c\omega}{k}$ or $\alpha = \tan^{-1}\left[2\xi \dfrac{\omega}{\omega_n}\right]$ ... 4.15

Equation 4.15 can be compared to a system excited by an external harmonic force, the steady-state amplitude $X = \dfrac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$,

where $F_0 = Y\sqrt{k^2 + (c\omega)^2}$ from last equation

$$\therefore \qquad X = \frac{Y\sqrt{k^2 + c\omega^2}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \text{ or } \frac{X}{Y} = \frac{\sqrt{k^2 + (c\omega)^2}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \qquad \text{... 4.16}$$

Equation 4.16 can be expressed in nondimensional form by dividing numerator and denominator by 'k'.

$$\frac{X}{Y} = \frac{\sqrt{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}} \quad \therefore \text{T R} = \frac{F_{tr}}{F_0} = \frac{X}{Y} = \frac{\sqrt{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}} \quad \text{...4.17}$$

and $\tan \beta = \frac{c\omega}{k}$ or $\beta = \tan^{-1}\left[2\xi \frac{\omega}{\omega_n}\right]$ or it can be written as $\beta = \tan^{-1}[2\xi r]$

where $r = \frac{\omega}{\omega_n}$ frequency ratio and

$$\text{phase lag is given by } (\alpha - \beta) = \tan^{-1}\left(\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)}\right) - \tan^{-1}\left[2\xi\left(\frac{\omega}{\omega_n}\right)\right] \quad \text{... 4.18}$$

Figure 4.10(a) shows the plots of **amplitude ratio** $\frac{X}{Y}$ against **frequency ratio** $\frac{\omega}{\omega_n}$ for various values of damping factor and Fig. 4.10(b) shows the plots of **phase angle** $(\alpha - \beta)$ against the **frequency ratio** $\frac{\omega}{\omega_n}$ for various values of damping factor. From the figure, it is seen that

(i) when $\frac{\omega}{\omega_n} < \sqrt{2}, \frac{X}{Y} > 1$ (ii) when $\frac{\omega}{\omega_n} > \sqrt{2}, \frac{X}{Y} < 1$

(iii) when $\frac{\omega}{\omega_n} = \sqrt{2}, \frac{X}{Y} = 1$ (iv) when $\frac{\omega}{\omega_n} = 1, \frac{X}{Y} = \infty$

2. Relative motion In Absolute motion, we assumed that the displacement of the mass 'm' relative to the support is 'z'.

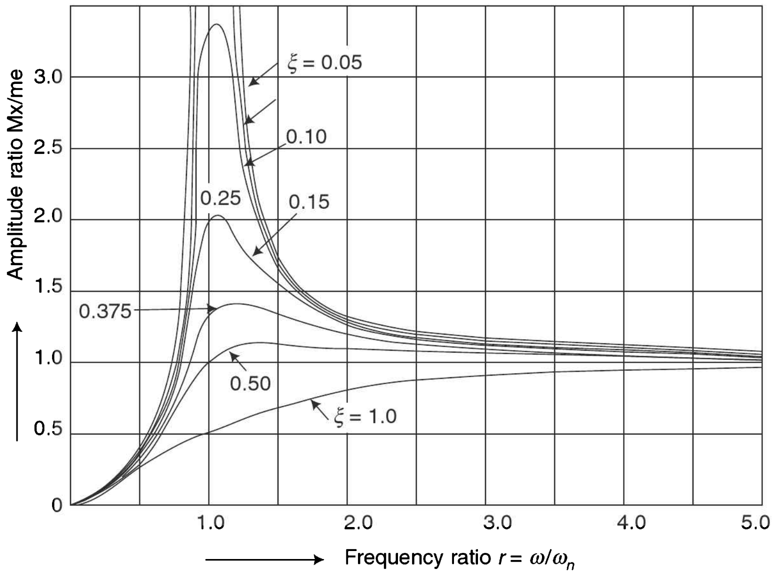


Fig. 4.10 (a) *Amplitude ratio verses frequency ratio*

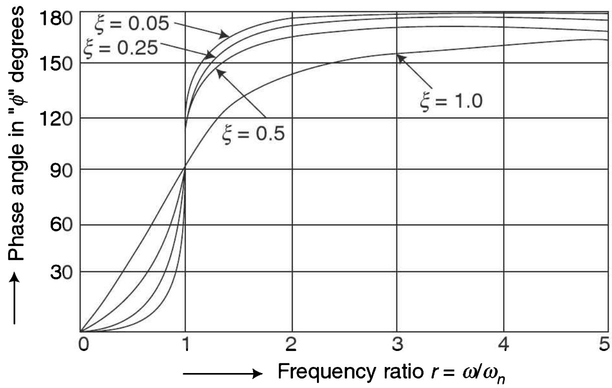


Fig. 4.10 (b) *Phase angle versus frequency ratio*

Then this can be written as

$$z = (x - y), \quad \dot{z} = (\dot{x} - \dot{y}), \quad \ddot{z} = (\ddot{x} - \ddot{y})$$

Substituting these valves in Eq. 4.14 and simplifying, we get

$$\therefore \qquad m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \qquad \text{...4.19}$$

We have $\quad y = Y \sin \omega t, \quad \dot{y} = Y\omega \cos \omega t \quad \therefore \ddot{y} = -Y\omega^2 \sin \omega t$

Hence Eq. 4.19 can be written as

$$m\ddot{z} + c\dot{z} + kz = m\omega^2 y \sin \omega t \qquad \text{...4.20}$$

Eq. 4.20 is in the same form of rotating and reciprocating unbalance Eq. 4.10

Therefore, the steady-state relative amplitude is

$$Z = \frac{Y\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \text{ or } \frac{Z}{Y} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}} \qquad \text{...4.21}$$

where $\frac{Z}{Y}$ is called relative displacement transmissibility

and phase angle is given by $\phi = \tan^{-1} \dfrac{2\xi\left(\frac{\omega}{\omega_n}\right)}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}$. ... 4.22

ENERGY DISSIPATED BY DAMPING

During forced vibration with viscous damping, energy is continuously absorbed by the damper so that power must be supplied to maintaining steady-state condition. So there is a need to evaluate the magnitude of power required and to see how it changes with the variables.

Work done by the force '*F*' during an interval of time when the body moves through a displacement '*dx*' is given by $dw = F\,dx$.

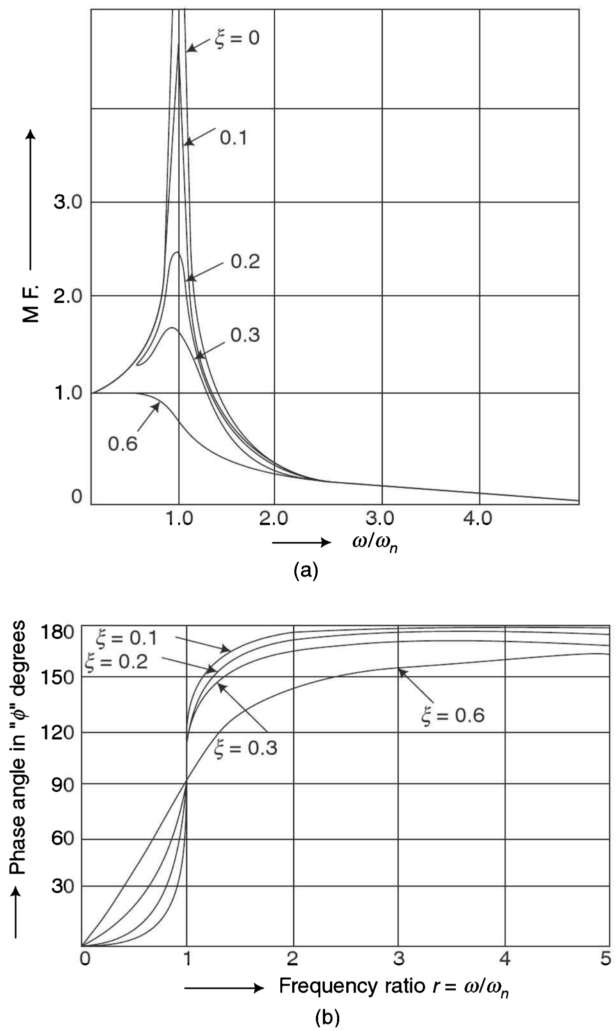


Fig. 4.11 *(a) Magnification factor versus frequency ratio (b) Phase angle versus frequency ratio*

$= F \cdot \frac{dx}{dt} \times dt$. Over a period of one cycle displacement, 'ωt' varies from 0 to 2π. Therefore, 't' varies from 0 to $\frac{2\pi}{\omega}$.

$\therefore$ work done per cycle is given by $w = \int_0^{2\pi/\omega} F \cdot \frac{dx}{dt} \cdot dt$ but $x = X \sin(\omega t - \phi)$

$$\therefore \quad \frac{dx}{dt} = X\omega \cos(\omega t - \phi), \text{ and } F = F_0 \sin \omega t$$

$$w = \int_0^{2\pi/\omega} (F_0 \sin \omega t)(X\omega \cos(\omega t - \phi))dt \qquad \text{... 4.23,}$$

$$\omega = \pi F_0 \times \sin \phi$$

where X = Amplitude of vibratory motion

F_0 = Amplitude of vibrating force

ϕ = Phase angle by which the motion lags the force

The maximum work is absorbed when the face angle ϕ is 90° and when $\frac{\omega}{\omega_n} = 1$ and $\sin \phi = \sin 90° = 1$.

Therefore, work done per cycle or energy dissipated per cycle = $\pi(c\omega X)\, X$.

Energy dissipated per cycle = $\pi\, c\omega X^2$. ...4.24

FORCED VIBRATION WITH COULOMB DAMPING

As we know from Chapter 3, Sec. 3.3.1, Case (b) on different types of damping, Coulomb damping or dry friction damping is caused by friction between the surfaces that are dry or having insufficient lubrication. When a body slides on a dry surface, the force of resistance between the surfaces or the frictional force is proportional to the normal load. This damping is called Coulomb damping and is shown in Fig. 4.12.

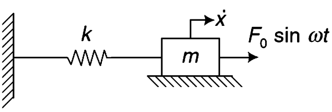


Fig. 4.12 *Coulomb damping*

When a single-degree-of-freedom system under Coulomb damping or dry friction damping is subjected to a harmonic force of '$F_0 \sin \omega t$' the differential equation of a motion is written as

$$m\ddot{x} + kx \pm \mu R_N = F_0 \sin \omega t \qquad ...4.25$$

In Eq. 4.25, the sign of friction force ($\pm\, \mu R_N$) is positive, when the body moves from left to right and it is vice versa as we know. In small values of Coulomb damping, the exact solution for small damping force is small so that the motion is continuous. On the other hand, in case of high value of Coulomb damping force, the motion does not remain continuous. Also, if dry friction force is small compared to harmonic force, an approximate solution is necessary. By determining an equivalent viscous damping 'c_{eq}' in case of forced vibration with Coulomb damping, the means of energy absorbed per cycle is same in both the cases.

Let 'X' be the amplitude of steady-state vibration and 'F' be the constant frictional force. Then the energy absorbed per cycle is $4FX$. For the similar amplitude of vibration, the energy absorbed per cycle for the case of equivalent viscous damping from Eq. 4.24 is

energy dissipated per cycle = $\pi\, c\omega X^2 = \pi\, c_{eq}\, \omega X^2$ since $c = c_{eq}$

Therefore, equating the two equations, $4FX = \pi\, c_{eq}\, \omega X^2, = c_{eq} = \dfrac{4F}{\pi\, \omega X}$ ...4.26

We have known that in case of viscous damping the steady-state amplitude is given by Eq. 4.5 in Sec. 4.2.

VIBRATION ISOLATION AND FORCE TRANSMISSIBILITY

When an unbalanced machine is mounted on the foundation, the vibration of the machine will be transmitted to the foundation. In order to minimise the transmission of forces to the foundation, machines are often mounted on springs and dampers as shown in the Fig. 4.15(a) The vibratory force transmitted to the foundation is by springs and dampers because these are the only connections. At any instant give a displacement 'x' to the mass 'm', and the FBD is as shown in Fig. 4.15(b).

Applying Newton's second law of motion to mass 'm', $\Sigma F = m\ddot{x}$.

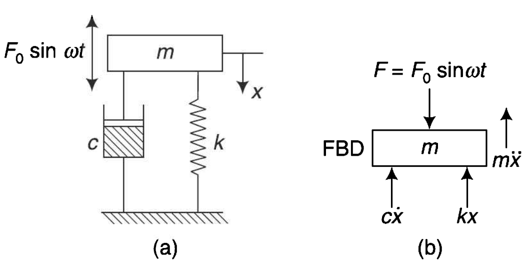


Fig. 4.15 *Vibration isolation and transmissibility*

$\therefore \quad F - c\dot{x} - kx = m\ddot{x} \quad m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t$ ...4.48

Since $\quad F = F_0 \sin \omega t$

this is a second-order nonhomogeneous differential equation of motion. The solution is

$$x = x_c + x_p$$

where x_c = Complementary function (already discussed in **Sec. 4.2** case '*a*')

x_p = Particular integral (steady state response already discussed in **Sec. 4.2** case '*b*')

Considering the steady-state response or to find the particular integral 'x_p'

Let $x_p = x = X \sin(\omega t - \phi)$ [$\because F$ is the external force which is a sinusoidal one]

$$\dot{x} = \omega X \cos(\omega t - \phi) = \omega X \sin\left(\frac{\pi}{2} + \omega t - \phi\right), \quad \ddot{x} = -\omega^2 X \sin(\omega t - \phi)$$

Substituting these value in Eq. 4.13,

$$m[-\omega^2 X \sin(\omega t - \phi)] + c\left[\omega X \sin\left(\frac{\pi}{2} + \omega t - \phi\right)\right] + kX \sin(\omega t - \phi) = F_0 \sin \omega t$$

Rearranging the terms,

$$F_0 \sin \omega t - kX \sin(\omega t - \phi) - cX\omega \sin\left(\omega t - \phi + \frac{\pi}{2}\right) + m\omega^2 X \sin(\omega t - \phi) = 0 \quad ...4.49$$

These forces can be vectorially represented as shown in Fig. 4.16(a),

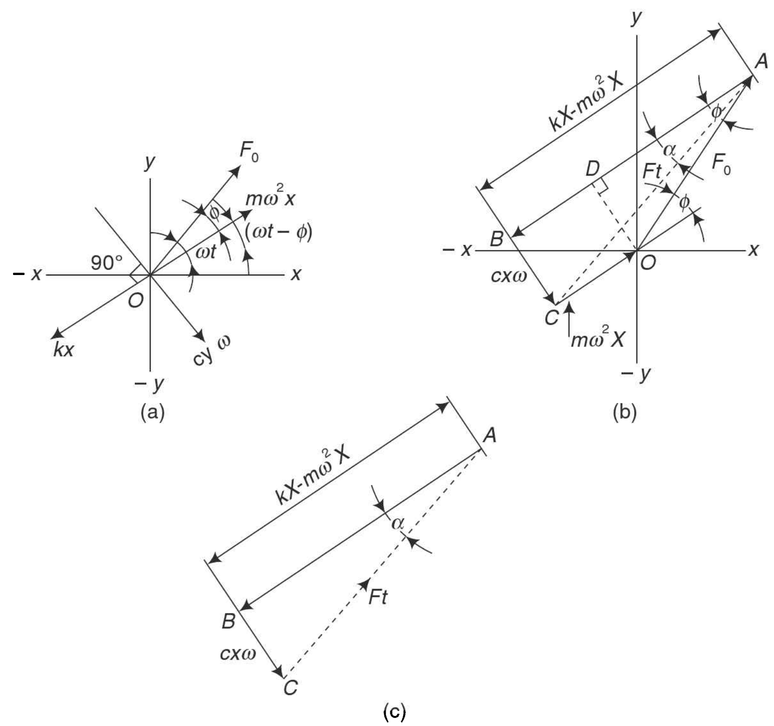


Fig. 4.16 *Force and vector polygon*

$$\overrightarrow{AC} = F_t = \sqrt{(kX)^2 + (cX\omega)^2} \quad \text{But } \overrightarrow{OA} = F_0, \overrightarrow{AB} = kX, \overrightarrow{BC} = C\omega X, \overrightarrow{CO} = mX\omega^2$$

The force transmitted to the foundation denoted by 'F_t' by joining 'AC' is the vectorial sum of the spring force and the damping force and it is shown in the Fig. 4.16(b) force polygon.

The *transmissibility ratio* or *transmissibility* is defined as the ratio of the force transmitted to the foundation 'F_t' through elastic supports to the force transmitted to the foundation through rigid supports 'F_0' (exiting force) **(See How to Draw the Vector and Force Polygon.)**

$$\therefore \text{ transmissibility TR} = \frac{F_1}{F_0}$$

From right-angled triangle OAD in Fig. 4.16(b),

$$OA^2 = AD^2 + DO^2 = (AB - BD)^2 + BC^2$$

$$\therefore \quad F_0^{\,2} = (kX - mX\omega^2)^2 + (cX\omega)^2, F_0^{\,2} = X^2\,[(k - m\omega^2)^2 + (c\omega)^2]$$

$$X^2 = \frac{F_0^2}{(k - m\omega^2)^2 + (c\omega)^2}, \quad X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

$$\therefore \quad X = \frac{F_0}{k\sqrt{\left[\left(1 - \frac{m}{k}\omega^2\right)^2 + \left(\frac{c}{k}\omega\right)^2\right]}}$$

$$\text{But} \quad \frac{m}{k} = \frac{1}{\omega_n^2} \text{ and } \frac{c}{k} = \frac{2\xi}{\omega_n}, \text{ and let } \frac{\omega}{\omega_n} = r.$$

$$\therefore \quad X = \frac{F_0}{k\sqrt{[(1 - r^2)^2 + (2\xi r)^2}} \qquad \text{...4.50}$$

$$\therefore \quad F_0 = kX\sqrt{[(1 - r^2)^2 + (2\xi r)^2} \qquad \text{...4.51}$$

From triangle ABC in Fig. 4.16(c),

$$F_t^2 = (kX)^2 + (cX\omega)^2, F_t^2 = k^2X^2\left[1 + \left(\frac{c}{k}\omega\right)^2\right], F_t = kX\sqrt{1 + (2\xi r)^2} \qquad \text{...4.52}$$

From Eq. 4.49 and Eq. 4.50,

$$\therefore \quad \text{TR} = \frac{F_t}{F_0} = \frac{kX\sqrt{1 + (2\xi r)^2}}{kX\sqrt{[(1 - r^2)^2 + (2\xi r)^2}} \qquad \text{...4.53}$$

$$\frac{F_t}{F_0} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{[(1 - r^2)^2 + (2\xi r)^2}} \qquad \text{...4.54}$$

The phase angle $\phi = \tan^{-1}\left[\frac{2\xi r}{1 - r^2}\right]$

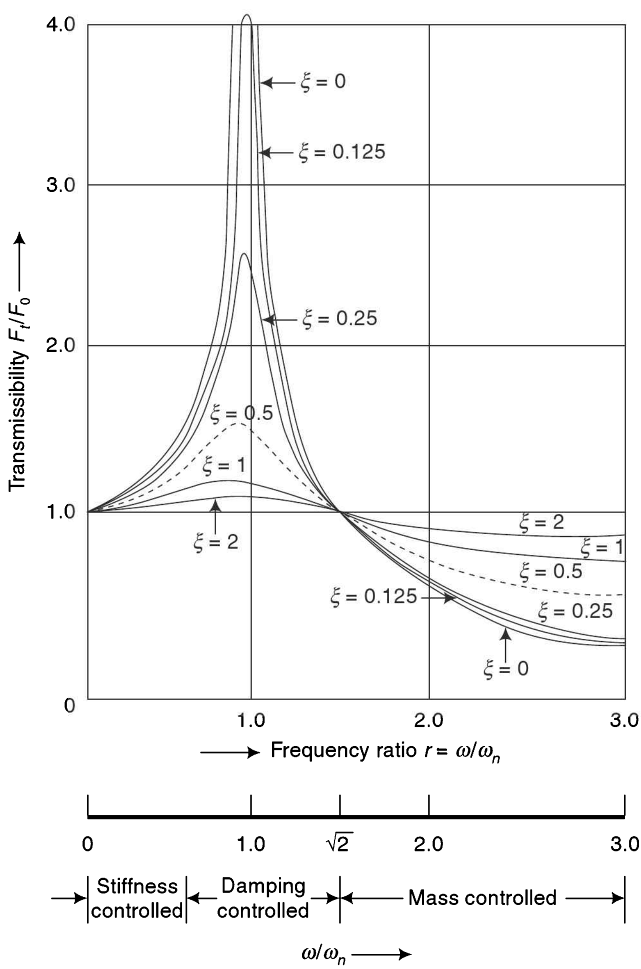
$$\therefore \qquad \phi = \tan^{-1}\left[\frac{2\xi r}{1 - r^2}\right] \qquad \text{...4.55}$$

and $\qquad \tan\alpha = \dfrac{cX\omega}{kX}, \; \alpha = \tan^{-1}(2\xi r).$

The angle of lag is given as a $(\phi - \alpha)$ $\quad \therefore \tan^{-1}\left[\dfrac{2\xi r}{1 - r^2}\right] - \tan^{-1}(2\xi r)$ ... 4.56

The equations 4.54 and 4.55 indicate the transmissibility and phase lag of transmitted force from the impressed force and can be plotted as shown in Fig. 4.17(a) and (b) for various values of damping factors.

1. Discussion on $\dfrac{F_t}{F_0}$ Versus $\dfrac{\omega}{\omega_n}$.



(a) Transmissibility versus frequency ration for various amount of damping factors

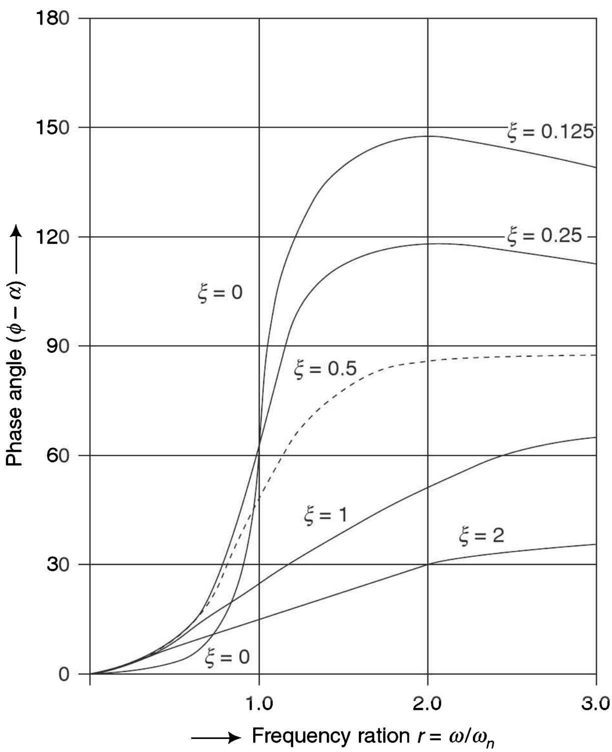


Fig. 4.17(b) *Phase angle frequency ratio for various amount of damping factors*

Case (i) When *r* = 0 in Eq. 4.54 $\dfrac{F_t}{F_0} = \dfrac{\sqrt{1 + (2\xi r)^2}}{\sqrt{[(1 - r^2)^2 + (2\xi r)^2}}.$

$$\frac{F_t}{F_0} = \frac{\sqrt{1 + 0}}{\sqrt{1 + 0}}, \quad \frac{F_t}{F_0} = 1, \text{ which is independent of the damping ratio '}\xi\text{'}$$

Case (ii) When *r* = 1 (Resonance, i.e. $\omega = \omega_n$) in Eq. 4.54

$$\frac{F_t}{F_0} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{[(1 - r^2)^2 + (2\xi r)^2}}$$

$$\frac{F_t}{F_0} = \frac{\sqrt{1 + (2\xi)^2}}{\sqrt{1 + (2\xi)^2}} = \frac{\sqrt{1 + (2\xi)^2}}{(2\xi)^2}, \text{ at } \xi = 0 \; \frac{F_t}{F_0} = \infty$$

i.e. at resonance and at damping ratio = 0, $\dfrac{F_t}{F_0} = \infty$

Force transmitted to the foundation with elastic supports = ∞, F_t can be brought down to nominal value by introducing damping into the system.

Case (iii) When *r* >> 1 ($\omega >> \omega_n$) in Eq. 4.57

$$\frac{F_t}{F_0} = \frac{\sqrt{1 + (2\xi r)^2}}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

i.e. $r^2 >> 1 \quad \therefore \frac{1}{r^2} <<< 1$, i.e. $1/r^2 \approx 0$ and $1/r \approx 0$

$$\frac{F_t}{F_0} = \frac{\sqrt{r^2\left(\frac{1}{r^2} + (2r)^2\right)}}{\sqrt{r^2\left(\left(\frac{1}{r^2} - 1\right)^2 + (2r)^2\right)}}, \quad \frac{F_t}{F_0} = \frac{\sqrt{r^2\left(\frac{1}{r^2} + (2\xi)^2\right)}}{\sqrt{r^2\left(\left(\frac{1}{r^2} - 1\right)^2 + (2\xi)^2\right)}}$$

$$\frac{F_t}{F_0} = \frac{2\xi}{\sqrt{1 + (2\xi)^2}} \text{ for } \xi << 1,\ \xi^2 \approx 0 \text{ and } \xi \approx 0.$$

$\therefore$ for $\xi << 1, \frac{F_t}{F_0} \approx 0$; for $\xi >> 1 \ \frac{F_t}{F_0} \approx 1$

MECHANICAL VIBRATIONS

III B.TECH – I SEM

UNIT – IV

Two Degree of Freedom Systems

Vibrations of undamped system , torsional system , damped free vibrations, forced harmonic vibration , coordinate coupling and principal coordinates , torsional vibration absorber , centrifugal pendulum absorber

TWO-DEGREE-FREEDOM SYSTEMS

INTRODUCTION

Systems that require two independent coordinates to specify the system configuration at any instant are called **'two-degree-freedom systems'**. In such a system there are two masses which have two equations of motion, treated as coupled differential equations. Each mass will have its own natural frequency. Sometimes nonharmonic motion of the masses makes the system more complicated for solving problems.

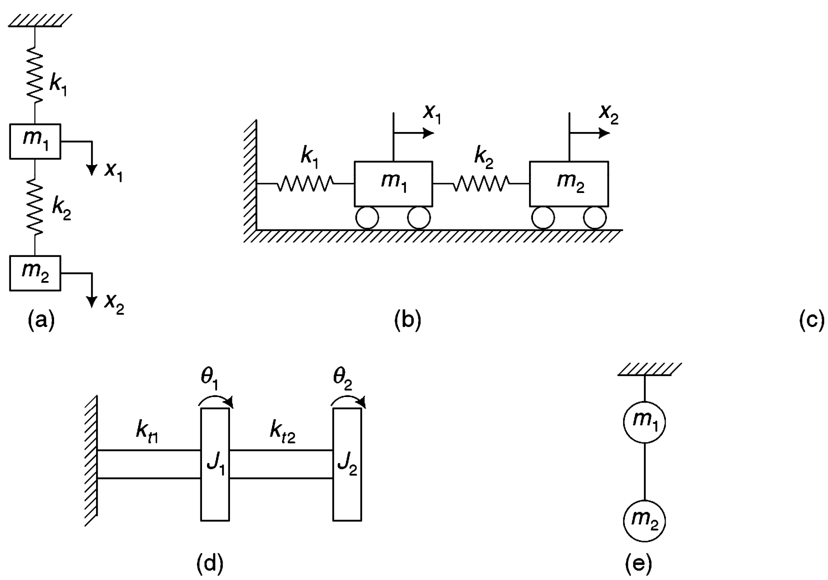


Fig. 6.1 *Two-degree-freedom systems*

Example In a spring-mass system, $k_1 - m_1$ and $k_2 - m_2$ are shown in Fig. 6.1(a) and Fig. 6.1(b). if the masses 'm_1' and 'm_2' are constrained to move vertically or horizontally (linear displacements), two independent coordinates 'x_1' and 'x_2' are necessary to specify their positions at any instant. 'k_1' and 'k_2' are stiffnesses of the spring. Therefore, the given system is a two-degree-freedom system. Other examples are double pendulum and two-rotor system as shown in Fig. 6.1(c), (d), (e).

In Fig. 6.1(c), two masses of a simple pendulum are coupled together by means of a spring 'k'. Similarly, a shaft of torsional stiffness 'k_t' [Fig. 6.1(d)] has two rotors which can have angular displacements 'θ_1' and 'θ_2' independent of each other. In Fig. 6.1(e), two masses 'm_1' and 'm_2' of a simple pendulum are constrained to move vertically. Thus, it is a two-degree-freedom system.

In general, the number of degrees of a freedom system can be stated as the number of mass or masses in a system and multiplied by number of possible types of motion of each mass or masses.

PRINCIPAL MODE OF VIBRATION, OR NORMAL MODE OF VIBRATION

There are two equations of motion for a two-degree-freedom system, one for each mass. As a result, there are two natural frequencies, for a two-degree-freedom system. The natural frequencies are found by solving the frequency equation of an undamped system or the characteristic equation of a damped system.

When the masses of a system are oscillating in such a manner that they reach maximum amplitudes simultaneously and pass their equilibrium points simultaneously or all the moving parts of the system are oscillating in the same frequency and phase, such a mode of vibration is called **principal mode of vibration,** or **normal mode of vibration.**

If at the principal mode of vibration, the amplitude of one of the masses is considered equal to unity, the mode of vibration is then called **'normal mode of vibration'**, i.e. the amplitude ratio, X_2/X_1 = Principal mode of vibration, if $X_1 = 1$(unity). Then the amplitude ratio, $X_2/1$ = Normal mode of vibration.

In case of two-degrees-freedom system, masses will vibrate in two different modes called '**principal modes**'. If masses 'm_1' and 'm_2' shown in Fig. 6.1(a) are vibrating in phase, such a mode of vibration is called **first principal mode**. When the masses 'm_1' and 'm_2' are vibrating in the opposite phase, such a mode of vibration is called **second principal mode** of vibration.

$\frac{X_2}{X_1}$ = Principal mode, and $\frac{X_2}{1} = \frac{1}{X_1}$ = Normal mode.

In first principal mode, $\frac{X_2}{X_1}$ or $\frac{X_1}{X_2}$ is positive, $X_1 \uparrow$ and $X_2 \uparrow$

In first principal mode, $\frac{X_2}{X_1}$ or $\frac{X_1}{X_2}$ is negative, $X_1 \downarrow$ and $X_2 \uparrow$

ORTHOGONALITY PRINCIPLE

The principal modes or normal modes of vibration for systems having two or more degrees of freedom are orthogonal. This is known as orthogonality principle.

This is an important property while finding the natural frequencies.

For a two-degree-freedom system, orthogonality principle can be written as

$$m_1A_1A_2 + m_2B_1B_2 = 0,$$

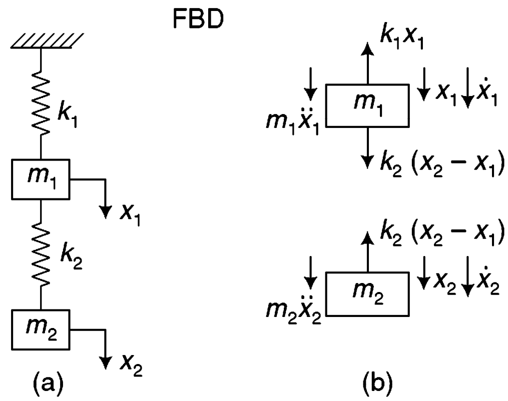


Fig. p-6.1 *Two-degree linear spring-mass system*

Solution Now at any instant, give displacement 'x_1' to the mass 'm_1' and 'x_2' to the mass 'm_2' to Fig. p-6.1(a). The FBD is as shown in Fig. p-6.1(b).

Applying Newton's second law of motion to mass 'm_1', assuming that $x_2 > x_1$

$$\Sigma F = m\,a$$

$$\therefore \quad k_2(x_2 - x_1) - k_1x_1 = m_1\ddot{x}_1$$

$$\therefore \quad m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) = 0$$

$$\therefore \quad m_1\ddot{x}_1 + k_1x_1 - k_2x_2 + k_2x_1 = 0$$

$$\therefore \quad m_1\ddot{x}_1 + (k_1 + k_2)\,x_1 - k_2x_2 = 0$$

But the given values of $m_1 = m,\ k_1 = 2k,\ k_2 = k.$

$$\therefore \quad m\ddot{x}_1 + (2k + k)\,x_1 - kx_2 = 0,\ m\ddot{x}_1 + 3kx_1 - kx_2 = 0 \qquad ...6.1$$

This is the differential equation of motion of the mass 'm_1'.

Again apply Newton's second law of motion to the mass 'm_2'.

$$\Sigma F = m\,a$$

$$\therefore \quad -k_2(x_2 - x_1) = m_2\ddot{x}_2$$

$$\therefore \quad m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0$$

$$\therefore \quad m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0$$

But the given values of $k_1 = 2k,\ m_2 = 2m,\ k_2 = k$

$$\therefore \quad 2m\ddot{x}_2 + kx_2 - kx_1 = 0 \qquad ...\ 6.2$$

This is the differential equation of motion of the mass 'm_2'.

Assume that the motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$x_1 = A \sin \omega t \qquad x_2 = B \sin \omega t$$

$$\dot{x}_1 = \omega A \cos \omega t \qquad \dot{x}_2 = \omega B \cos \omega t$$

$$\ddot{x}_1 = -\omega^2 A \sin \omega t \quad \ddot{x}_2 = -\omega^2 B \sin \omega t$$

Using the values of x_1, x_2 and $\ddot{x}_1$ in Eq. 6.1, we get

$$m(-A\omega^2 \sin \omega t) + 3kA \sin \omega t - kB \sin \omega t = 0$$

$$-m\omega^2 A \sin \omega t + 3\,kA \sin \omega t = kB \sin \omega t$$

$$A \sin \omega t\,(3k - m\omega^2) = kB \sin \omega t,\ A\,(3k - m\omega^2) = kB$$

The amplitude ratio $\therefore \dfrac{A}{B} = \dfrac{k}{3k - m\omega^2}$ .. 6.3

Again using the values of x_1, x_2 and $\ddot{x}_2$ in Eq. 6.2, we get

$$2m(-\omega^2 B \sin \omega t) + k(B \sin \omega t) - k(A \sin \omega t) = 0$$

$$-2m\omega^2 B \sin \omega t + kB \sin \omega t = kA \sin \omega t$$

$$B \sin \omega t\,(k - 2m\omega^2) = kA \sin \omega t,\ B(k - 2m\omega^2) = kA$$

The amplitude ratio, $\dfrac{A}{B} = \dfrac{k - 2m\omega^2}{k}$ ...6.4

From equations 6.3 and 6.4,

$$\frac{k}{3k - m\omega^2} = \frac{k - m\omega^2}{k}$$

$$(3k - m\omega^2)\,(k - 2m\omega^2) = k^2$$

$$3k^2 - 2m\omega^2 \times 3k - m\omega^2 k + 2m^2\omega^4 = k^2,\ 2m^2\omega^4 - 7km\omega^2 + 2k^2 = 0$$

$$\omega^4 - \frac{7k}{2m}\omega^2 + \frac{k^2}{m^2} = 0$$

This is a quadratic equation in ω^2, where roots are given by

$$\omega^2 = \frac{+\dfrac{7k}{2m} \pm \sqrt{\left(\dfrac{7k}{2m}\right)^2 - \dfrac{4k^2}{m^2}}}{2},\ \omega^2 = \frac{7k}{4m} \pm \sqrt{\frac{\dfrac{49k^2}{4m^2} - \dfrac{4k^2}{m^2}}{4}}$$

$$\omega^2 = \frac{7k}{4m} \pm \sqrt{\frac{49k^2 - 16k^2}{16m^2}},\ \omega^2 = \frac{7k}{4m} \pm \sqrt{\frac{33k^2}{16m^2}}$$

$$\omega^2 = \frac{7k}{4m} \pm \frac{5.74k}{4m},\ \omega_{1n}^2 = \frac{7k}{4m} - 5.74k/4m,\ \omega_{2n}^2 = \frac{7k}{4m} + \frac{5.74k}{4m}$$

$$\omega_{1n}^2 = 0.315\,\frac{k}{m}\ \omega_{2n}^2 = 3.185\,\frac{k}{m},\ \omega_{1n} = 0.56\sqrt{\frac{k}{m}}\ \text{rad/s},\ \omega_{2n} = 1.78\sqrt{\frac{k}{m}}\ \text{rad /s}$$

where ω_{1n} and ω_{2n} are the first and second natural frequencies respectively.

To draw the mode shapes

(i) First mode shape. From Eq. 6.3

$$\frac{A}{B} = \frac{k}{3k - m\omega^2}$$

At $\quad \omega^2 = \omega_{1n}^2 = 0.315 \dfrac{k}{m}, \qquad \dfrac{A}{B} = \dfrac{k}{3k - m \times 0.315 \dfrac{k}{m}}$

$\therefore \quad \dfrac{A}{B} = \dfrac{1}{2.69}$, i.e. at $A = 1$, $B = 2.6$ [Fig. p-6.1(c)]

(ii) Second mode shape. From Eq. 6.4, $\dfrac{A}{B} = \dfrac{k - 2m\omega^2}{k}$

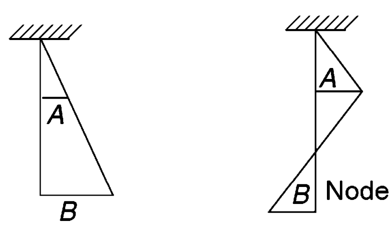
At $\quad \omega^2 = \omega_{2n}^2 = 3.185 \dfrac{k}{m}$

At $\quad A = 1$

$\therefore \quad \dfrac{A}{B} = \dfrac{k - 2m \times 3.185 \dfrac{k}{m}}{k} = \dfrac{k - 6.37k}{k}$

$\therefore \quad \dfrac{A}{B} = -5.37$

$\therefore \quad A = -5.37\, B$ [Fig. p-6.1(d)]



(c) Same phase (d) Out of phase

Fig. p-6.1 *Mode shape*

In the first mode of Fig. p-6.1(c) the full spring moves to the right side of the mean line as it is 'same phase'.

Whereas in the second mode of Fig. p-6.1(d), the second spring crosses the mean line as it is 'out of phase'. The crossed point is called **'node'** point, i.e. there is no displacement at that point.

SEMIDEFINITE SYSTEM OR DEGENERATING SYSTEM

This is defined as a system where one natural frequency is equal to zero. This is also known as a degenerate system. Consider the system to represent two masses 'm_1' and 'm_2' and with a coupling spring 'k' as shown in Fig. 6.2(a).

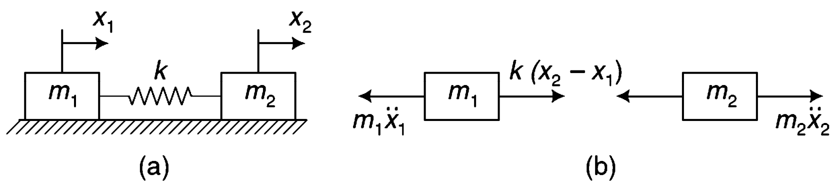


Fig. 6.2 *Semidefinite system*

Now at any instant, give displacement 'x_1' to the mass 'm_1' and 'x_2' to the mass 'm_2' to the Fig. 6.2(a). The FBD is as shown in Fig. 6.2(b).

Assuming that $x_2 > x_1$ or $x_1 > x_2$ also can be taken, but $x_2 > x_1$ is easy to writing down the differential equations.

Apply Newton's second law of motion to the mass 'm_1', i.e. $\Sigma F = ma$

$$m_1 \ddot{x}_1 = -k(x_1 - x_2), \quad m_1 \ddot{x}_1 + k(x_1 - x_2) = 0 \qquad ...6.61$$

Similarly apply Newton's second law of motion for the mass 'm_2',

$$m_2\ddot{x}_2 = -k(x_2 - x_1), \quad m_2\ddot{x}_2 + k(x_2 - x_1) = 0 \qquad ...6.62$$

Assume that motion is periodic and is composed of harmonic motions of varies amplitudes and frequencies. Let one of these components be,

$$x_1 = A \sin \omega t, \qquad x_2 = B \sin \omega t$$

$$\dot{x}_1 = A\omega \cos \omega t, \qquad \dot{x}_2 = B\omega \cos \omega t$$

$$\ddot{x}_1 = -A\omega^2 \sin \omega t, \qquad \ddot{x}_2 = -B\omega^2 \sin \omega t$$

Substituting these values in equations 6.61 and 6.62, we get

$$(k - m_1\omega^2) A - kB = 0 \qquad ...6.63$$

$$-kA + (k - m_2\omega^2) B = 0 \qquad ...6.64$$

The frequency equation is obtained by equating to zero the determinants of the coefficient 'A' and 'B' are

$$\begin{vmatrix} (k - m_1\omega^2) & -k \\ -k & (k - m_2\omega^2) \end{vmatrix} = 0, \; (k - m_1\omega^2)\,(k - m_2\omega^2) - (-k)\,(-k) = 0$$

$$k^2 - km_2\omega^2 - km_1\omega^2 + m_1m_2\omega^4 - k^2 = 0, \; m_1m_2\omega^4 - km_2\omega^2 - km_1\omega^2 = 0$$

$\omega^2\,[m_1m_2\omega^2 - k\,(m_1 + m_2)] = 0 \quad \therefore\; \omega_1 = 0$, as one of their natural frequencies is equal to zero as the statement of the semidefinite system and $\omega_2 = \sqrt{\dfrac{k(m_1 + m_2)}{m_1m_2}}$ rad/s.

Mode shapes Dividing Eq. 6.64 by Eq. 6.63, we get

$$-kA + (k - m_2\omega^2)\,B/(k - m_1\omega^2)\,A - kB, \quad \frac{A}{B} = -\,(m_1/m_2).$$

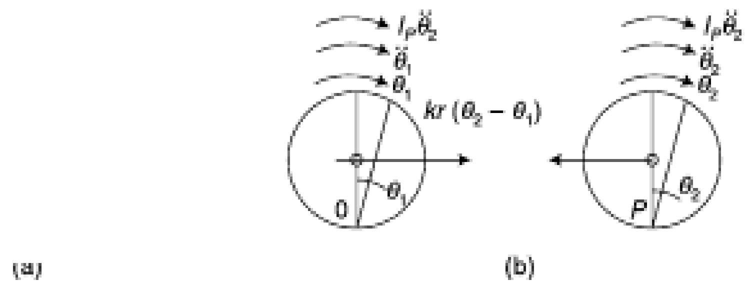


Fig. p-6.17 *Cylinder system*

Solution Let us at any instant give an angular displacement 'θ_1' to the first cylinder of mass 'm' and 'θ_2' to the second cylinder of mass 'm' as shown in Fig. p-6.17(a). Then FBD is as shown in Fig. p-6.17(b).

Applying Newton's second law of motion to the cylinder (1), let $\theta_2 > \theta_1$

$$\Sigma M_P = I_P\ddot{\theta}\,, -kr\,(\theta_2 - \theta_1)\,r = -I_P\ddot{\theta}\,, \quad -I_P\ddot{\theta} + kr^2\theta_2 - kr^2\theta_1 = 0$$

$$-\frac{3}{2}\,mr^2\ddot{\theta}_1 + kr^2\theta_2 - kr^2\theta_1 = 0, \quad -\frac{3}{2}\,m\ddot{\theta}_1 + k\theta_1 - k\theta_2 = 0 \quad ...6.65$$

This is the differential equation of motion for the cylinder (1).

Applying Newton's second law of motion to the cylinder (2),

$$\Sigma M_0 = I_0\ddot{\theta}, \quad kr\,(\theta_2 - \theta_1)\,r = -I_0\ddot{\theta}_2, \quad I_0\ddot{\theta}_2 + kr^2\theta_2 - kr^2\theta_1 = 0$$

$$\frac{3}{2}\,m\ddot{\theta}_2 + k\theta_2 - k\theta_1 = 0 \quad ...6.66$$

This is the differential equation of motion for the cylinder (2).

Assume that motion is periodic and is composed of harmonic motions of various amplitudes and frequencies. Let one of these components be,

$$\theta_1 = A \sin \omega t \qquad \theta_2 = B \sin \omega t$$

$$\ddot{\theta}_1 = -A\omega^2 \sin \omega t \qquad \ddot{\theta}_2 = -B\omega^2 \sin \omega t$$

Using the values of θ_1, θ_2, $\ddot{\theta}_1$ in Eq. 6.65,

$$\frac{3}{2} m (-A\omega^2) + kA - kB = 0, \quad A\left(k - \frac{3}{2} m\omega^2\right) = kB, \quad \frac{A}{B} = \frac{2k}{2k - 3m\omega^2} \qquad \text{...6.67}$$

Using the values of θ_1, θ_2, $\ddot{\theta}_2$ in Eq. 6.66,

$$\frac{3}{2} m(-B\omega^2) + kB - kA = 0, \quad B\left(k - \frac{3}{2} m\omega^2\right) = kA, \quad \frac{A}{B} = \frac{2k - 3m\omega^2}{2k} \qquad \text{...6.68}$$

From equations 6.67 and 6.68,

$$\frac{2k}{2k - 3m\omega^2} = \frac{2k - 3m\omega^2}{2k}, (2k - 3m\omega^2)^2 = (2k)^2, 2k - 3m\omega^2 = \pm 2k, 3m\omega^2 = 2k \pm 2k$$

$$\omega_{1n}^2 = 0, \quad \omega_{1n} = 0 \; \omega_{2n}^2 = \frac{4k}{3m}, \quad \omega_{2n} = \sqrt{\frac{4k}{3m}} \text{ rad/s}$$

Since one of the natural frequencies is zero, the system is a semidefinite system.

MECHANICAL VIBRATIONS

III B.TECH – I SEM

UNIT – V

Vibrations of Continuous Systems

Lateral Vibrations of springs, longitudinal vibrations of bars , transverse vibrations of beams.

VIBRATION OF CONTINUOUS SYSTEMS

Introduction

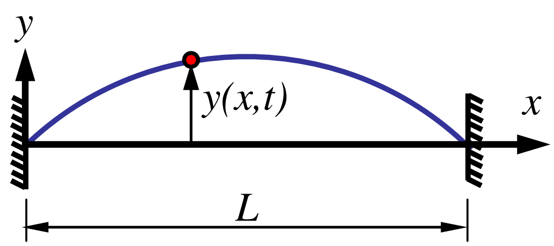
Models of vibratory systems can be divided into two broad classes, *lumped* and *continuous*, depending on the nature of the parameters. In the case of lumped systems, the components are discrete, with the mass assumed to be rigid and concentrated at individual points, and with the stiffness taking the form of massless springs connecting the rigid masses. The masses and springs represent the system parameters, and we refer to such models as *discrete* or *lumped-parameter* models. The motion of discrete systems is governed by *ordinary* differential equations. *Continuous* systems, on the other hand, differ from discrete systems in that the mass and elasticity are continuously distributed. Such systems are also known as *distributed-parameter* systems, and examples include strings, rods, beams, plates and shells. While discrete systems possess a finite number of degrees of freedom, continuous systems have an infinite number of degrees of freedom because we need an infinite number of coordinates to specify the displacement of every point in an elastic body. The displacement in this case depends on two independent variables, namely x and t. As a result, the motion of continuous systems is governed by partial differential equations to be satisfied over the entire domain of the system, subject to boundary conditions and initial conditions.

Although discrete systems and continuous system may appear entirely different in nature, the difference is more in form than concept. As a matter of fact, a certain physical system can be modeled either as discrete or as distributed, depending on the objectives of the analysis. It turns out that discrete and continuous systems are indeed closely connected, and thus it comes as no surprise that both systems possess natural frequencies and normal modes of vibration.

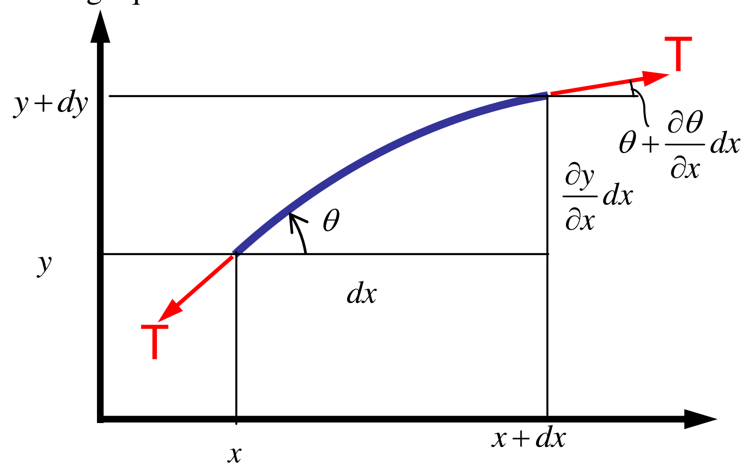
In this topic we will study the free and forced vibration of continuous systems. Emphasis will be placed on studying the vibration of taught strings, rods and beams. This covers a broad class of engineering applications, as many practical systems can be modeled by one or more of such elements in order to study the dynamic behavior

Vibration of Strings

The figure shows a fixed-fixed string of length L. The string is initially under tension T and the aim is to study the **transverse vibrations** denoted by the displacement $y(x,t)$, measured from the equilibrium position. It is assumed that both displacement and slope are small.



It is also assumed that the tension force in the string remains constant during vibration, which follows from the previous assumptions of small displacements. As in all continuous systems, the displacement variable depends on both the spatial (x) and temporal (t) coordinates. A free body diagram of a string element is shown below. Neglecting gravity effects, we can apply Newton's second law on the string element to obtain the governing equation of motion.



Applying $\uparrow \Sigma F_y = m \cdot a_y$ gives:

$$T\sin\left(\theta + \frac{\partial \theta}{\partial x}dx\right) - T\sin\theta = \rho dx \frac{\partial^2 y}{\partial t^2} \tag{1}$$

where ρ is the **mass per unit length** of the string. For small displacements, $\sin\theta \simeq \theta$, hence we obtain:

$$T\frac{\partial \theta}{\partial x} = \rho \frac{\partial^2 y}{\partial t^2} \tag{2}$$

But $\theta = \frac{\partial y}{\partial x}$ hence:

$$T \frac{\partial^2 y}{\partial x^2} = \rho \frac{\partial^2 y}{\partial t^2} \tag{3}$$

which can be written as:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \tag{4}$$

which is known as the one-dimensional wave equation and $c = \sqrt{\frac{T}{\rho}}$ is the velocity of wave propagation along the string. The wave equation is a partial differential equation, and the same form will be encountered in similar problems involving the dynamics of distributed-parameter models. The equation must be satisfied over the entire domain and is subject to **boundary conditions** as well as **initial conditions**. Accordingly, the problem posed is both a *boundary value problem (BVP)* and an *initial value problem (IVP)* from a mathematical point pf view.

We now seek the solution of the wave equation, which represents the variation of the transverse displacement at any point along the string and at any time for an arbitrary string that is set in motion by certain initial conditions and left to vibrate freely. This solution is emulated by the using the principle of separation of variables. In this way, the transverse displacement can be expressed as:

$$y(x,t) = Y(x) \cdot G(t) \tag{5}$$

It follows that:

$$\frac{\partial^2 y}{\partial x^2} = \frac{d^2 Y}{dx^2} \cdot G \tag{6}$$

and

$$\frac{\partial^2 y}{\partial t^2} = Y \cdot \frac{d^2 G}{dt^2} \tag{7}$$

Substitution into the equation of motion (4) yields:

$$\frac{d^2 Y}{dx^2} G = \frac{1}{c^2} Y \frac{d^2 G}{dt^2} \tag{8}$$

which can also be written as:

$$\frac{1}{Y} \frac{d^2 Y}{dx^2} = \frac{1}{c^2} \frac{1}{G} \frac{d^2 G}{dt^2} \tag{9}$$

It is noted that the left-hand-side (LHS) of the above equation depends **only** on the spatial variable x, whereas the RHS depends only on the temporal variable, t. In order to satisfy the equation, both sides of equation (9) must be equal to a constant. Let this constant be $-(\omega/c)^2$. A negative constant was conveniently selected because this choice leads to an oscillatory motion. The choice of a zero or positive constant does not yield a vibratory motion, and therefore must be excluded. For example, if a zero constant was chosen, this leads to:

$$\frac{1}{c^2}\frac{1}{G}\frac{d^2G}{dt^2}=0$$

or

$$\frac{d^2G}{dt^2}=0$$

whose solution is given by:

$$G=c_1t+c_2$$

which is rejected because it indicates a solution that increases linearly with time. It can be shown that the choice of a positive constant gives rise to two terms; one exponentially increasing and the other exponentially decreasing.

Adopting the negative constant choice, and substituting into the equation of motion gives:

$$\frac{1}{Y}\frac{d^2Y}{dx^2}=-(\omega/c)^2 \tag{10}$$

which can be written as:

$$\frac{d^2Y}{dx^2}+(\omega/c)^2\,Y=0 \tag{11}$$

Furthermore,

$$\frac{1}{c^2}\frac{1}{G}\frac{d^2G}{dt^2}=-(\omega/c)^2 \tag{12}$$

which can similarly be expressed as:

$$\frac{d^2G}{dt^2}+\omega^2G=0 \tag{13}$$

These have the general solutions:

$$Y(x)=A\sin(\omega/c)x+B\cos(\omega/c)x \tag{14}$$

And

$$G(t)=C\sin\omega t+D\cos\omega t \tag{15}$$

The 4 constants A, B, C and D are to be determined from the boundary conditions (BC's) and initial conditions (IC's). It also worthy to note that equation (14) defines the deformation shape, whereas equation (15) defines the motion to be harmonic in time. It becomes appropriate now to define the unknown constant ω as the natural frequency of the system, and $\left(\omega/c\right)$ as the wave number or spatial frequency. The general solution may then be expressed as:

$$y(x,t) = \left(A\sin(\omega/c)x + B\cos(\omega/c)x\right)\cdot\left(C\sin\omega t + D\cos\omega t\right) \tag{16}$$

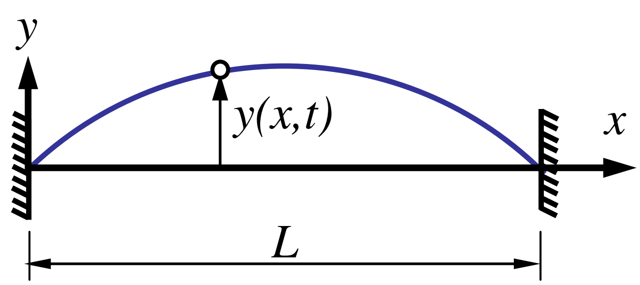
Alternatively, and after some algebraic manipulation, the above solution may also be written as:

$$\begin{aligned} y(x,t) = a_1\sin\left((\omega/c)x-\omega t\right) + a_2\cos\left((\omega/c)x-\omega t\right) + \\ a_3\sin\left((\omega/c)x+\omega t\right) + a_4\cos\left((\omega/c)x+\omega t\right) \end{aligned} \tag{17}$$

Once again, the solution must contain 4 unknown constants.

Example: Fixed-fixed string

Let us now consider the case of a string that is fixed at both ends, as shown.



The imposed boundary conditions indicate that the string displacement at both ends must be equal to zero, or:

$y(0,t)=0$ and $y(L,t)=0$. The general solution is:

$$\begin{aligned} y(x,t) &= Y(x)\cdot G(t) \\ &= \left(A\sin(\omega/c)x + B\cos(\omega/c)x\right)\cdot\left(C\sin\omega t + D\cos\omega t\right) \end{aligned}$$

Substitution of the first BC into the general solution gives:

$$0 = B\cdot\left(C\sin\omega t + D\cos\omega t\right)$$

which implies $B=0$. The general solution hence becomes:

$$y(x,t) = \left(A\sin(\omega/c)x\right)\cdot\left(C\sin\omega t + D\cos\omega t\right)$$

Substitution of the second BC into the solution gives:

$$0 = \left(A\sin(\omega L/c)\right)\cdot\left(C\sin\omega t + D\cos\omega t\right)$$

which implies:

$$\sin(\omega L/c) = 0$$

hence

$$\omega L/c = n\pi \quad , \quad n = 1,2,3,\ldots$$

The above equation is termed the **frequency equation** or **characteristic equation** of the system, as it gives values of the system natural frequencies. Clearly, the system possesses an *infinite number of natural frequencies*, as suggested earlier. Having obtained the natural frequencies, the solution at any frequency or mode is expressed by:

$$\begin{aligned} y_n(x,t) &= \left(A_n\sin(n\pi x/L)\right)\left(C_n\sin\omega_n t + D_n\cos\omega_n t\right) \\ &= Y_n(x)\cdot G_n(t) \end{aligned}$$

Therefore, at each natural frequency, there corresponds a certain **mode shape** or an **eigenfunction** defined by

$$Y_n(x) = A_n\sin\left(n\pi x/L\right)$$

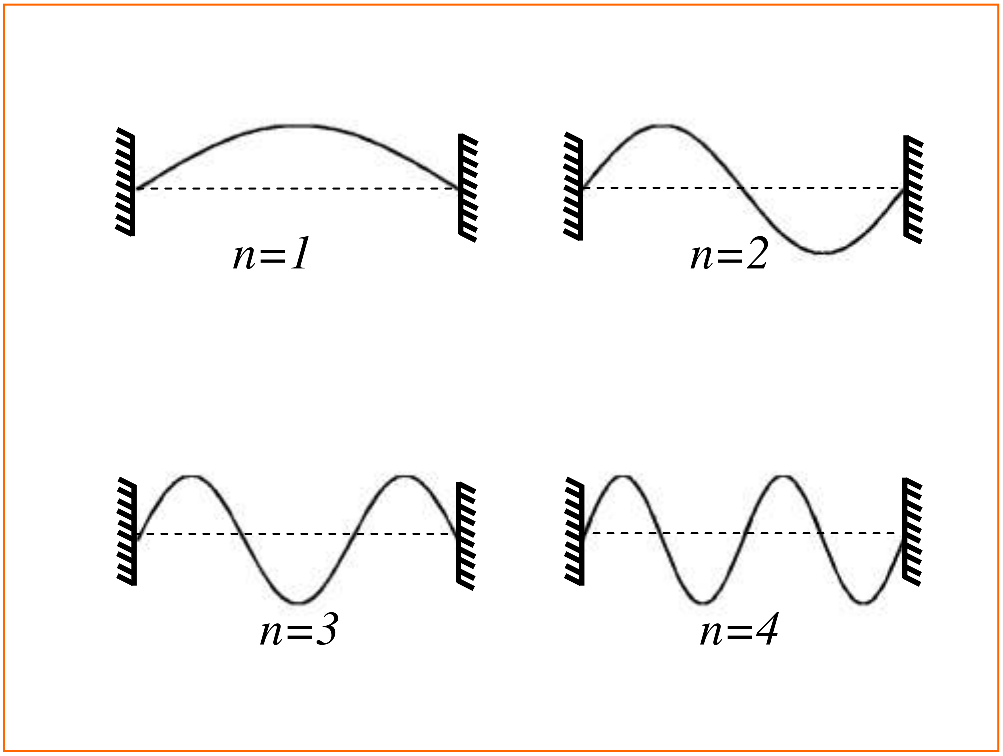
where each "*n*" represents a normal mode vibration with a natural frequency

$$\omega_n = \frac{n\pi c}{L}$$

and mode shape

$$Y_n(x) = A_n\sin\left(n\pi x/L\right)$$

where A_n are arbitrary constants. The figure below shows the first few modes of the string, as obtained from the above analysis.



The general solution is given by:

$$y_n(x,t) = \left(A_n \sin(n\pi x/L)\right)\left(C_n \sin\omega_n t + D_n \cos\omega_n t\right)$$

where the terms in the first bracket define the displacement pattern at mode "*n*", and the terms on the last bracket define a harmonic motion at the corresponding natural frequency. The free vibration solution is finally obtained as the sum of all modes of vibration, or:

$$y(x,t) = \sum_{n=1}^{\infty}\left(C_n \sin\omega_n t + D_n \cos\omega_n t\right)\sin(n\pi x/L)$$

where C_n, D_n are constants to be determined from the IC's.

The eigenfunctions can also be shown to possess an orthogonality property (see next section) which is given by:

$$\int_0^L Y_n(x)Y_m(x)dx = \begin{cases} 0 & n \neq m \\ h_n & n = m \end{cases}$$

For a fixed-fixed string, this becomes:

$$\int_0^L \sin(n\pi x/L)\sin(m\pi x/L)dx = \begin{cases} 0 & n \neq m \\ L/2 & n = m \end{cases}$$

In order to study the complete free vibration problem, the initial conditions must also be defined. Assume that the string is subjected to the following initial conditions:

$$y(x,0) = f(x) \quad , \quad \dot{y}(x,0) = g(x)$$

Substitution into the general solution gives:

$$f(x) = \sum_{n=1}^{\infty} D_n \sin(n\pi x/L)$$

and

$$g(x) = \sum_{n=1}^{\infty} C_n \omega_n \sin(n\pi x/L)$$

In order to obtain the constants C_n, D_n we multiply the above equations by $\sin(n\pi x/L)$ and integrate over the string length. By using the orthogonality codition, we get:

$$D_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)dx$$

and

$$C_n = \frac{2}{L\omega_n}\int_0^L g(x)\sin(n\pi x/L)dx$$

It thus follows that the constants C_n, D_n determine the **contribution** of each mode to the general solution. Now consider the special case where the initial conditions impose a displacement pattern that **coincides** with one of the natural modes, say mode "*k*", and that the initial velocity is zero. In this way, we have:

$$f(x) = \sin(k\pi x/L)$$

and

$$g(x) = 0$$

It follows that:

$$D_n = \begin{cases} 0 & k \neq n \\ 1 & k = n \end{cases}$$

and

$$C_n = 0$$

Therefore, the general solution reduces to:

$$y(x,t) = \sin(k\pi x/L)\cos\omega_k t$$

That is, the string vibrates in its "k^{th}" mode due to the fact that the initial conditions cause only the "k^{th}" mode to be excited. The continuous system in this special case behaves like a single DOF system. The same notion can be observed for a multi-DOF system. If, on the other hand, an impulse is given to the system, it can be shown that a wide spectrum of frequencies or modes are excited, and hence the system response will contain a summation of a large number of modes. In practice, the contribution of the higher modes is usually smaller, and the system response will be almost predominantly be described by the fundamental (i.e. first) mode, together with only a few higher modes.

Orthogonality of Eigenfunctions

Let us now prove the orthogonality condition for a fixed-fixed string. The same technique can be applied to other forms of boundary conditions. Recall Eq. (11) obtained previously:

$$\frac{d^2Y}{dx^2}+\left(\omega/c\right)^2 Y=0 \tag{18}$$

This equation must be satisfied at all natural frequencies or all normal modes of vibration. Consequently, at an arbitrary mode "*m*", we have ω_m, Y_m as the natural frequency and associated eigenfuction, respectively. Similarly, at mode "*n*", we have ω_n, Y_n. Let us first assume that these two modes are distinct. It follows from Eq. (18) that:

$$-\frac{d^2Y_m}{dx^2}=\left(\omega_m/c\right)^2 Y_m=0 \tag{19}$$

and

$$-\frac{d^2Y_n}{dx^2}=\left(\omega_n/c\right)^2 Y_n=0 \tag{20}$$

Multiplying Eq. (19) by Y_n and integrating over the string length yields:

$$-\int_0^L Y_n\frac{d^2Y_m}{dx^2}dx=\int_0^L\left(\omega_m/c\right)^2 Y_mY_n dx \tag{21}$$

The LHS of the above equation can be integrated by parts:

$$LHS=-\left[\left.Y_n\frac{dY_m}{dx}\right|_0^L-\int_0^L\frac{dY_m}{dx}\frac{dY_n}{dx}dx\right]=\int_0^L\frac{dY_m}{dx}\frac{dY_n}{dx}dx \tag{22}$$

and the first term vanishes due to the boundary conditions defined (both ends fixed). It follows that:

$$\int_0^L\frac{dY_m}{dx}\frac{dY_n}{dx}dx=\int_0^L\left(\omega_m/c\right)^2 Y_mY_n dx \tag{23}$$

Similarly, we can multiply equation (20) by Y_m and integrate over the string length. This yields:

$$\int_0^L\frac{dY_n}{dx}\frac{dY_m}{dx}dx=\int_0^L\left(\omega_n/c\right)^2 Y_mY_n dx \tag{24}$$

It is noted that the LHS of Eqs. (23) and (24) are identical, irrespective of the order of multiplication. Subtracting Eq. (24) from Eq. (23) gives:

$$0 = \frac{\omega_m^2 - \omega_n^2}{c^2} \int_0^L Y_m Y_n dx \tag{25}$$

But ω_m, ω_n are two distinct modes, i.e. $\omega_m \neq \omega_n$, therefore:

$$\int_0^L Y_m Y_n dx = 0 \quad , \quad m \neq n \tag{26}$$

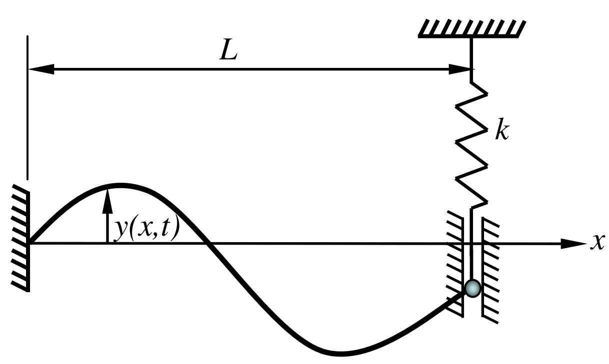
which proves the orthogonality condition for eigenfunctions of a fixed-fixed string. This condition also holds true for other types of BC's. Moreover, since eigenfunctions can be arbitrarily scaled (or *normalized*), we can write:

$$\int_0^L Y_m Y_m dx = h_m \quad , \quad m = n \tag{27}$$

where h_m is a constant.

Elastic or Inertial Attachments

Finally, let us now consider other forms of boundary conditions. Figure 6 shows a string that is fixed at one end and attached to a spring at the other. The boundary condition at the fixed end $x = 0$ is given by $y(0,t) = 0$. This is called a **geometric boundary condition**, because it describes a *specified* displacement. Such a condition is also known as an **essential** or **imposed** boundary condition.



Elastic attachments.

The boundary condition at the other end is not so obvious at first sight. Indeed it becomes appropriate to draw a free-body diagram of the string in order to investigate the force interaction. Such a free-body diagram is shown in Fig.7. At $x = L$ we need to balance forces in the vertical direction. Thus we have:

$$T \frac{\partial y}{\partial x}(L,t) = -ky(L,t) \tag{28}$$

This is called a **natural boundary condition** (also known as **dynamic** or **additional** boundary condition) as it describes forces and moments acting on the system. We can then proceed with the solution in the same way described above in order to obtain the natural frequencies, eigenfunctions and response to initial conditions.

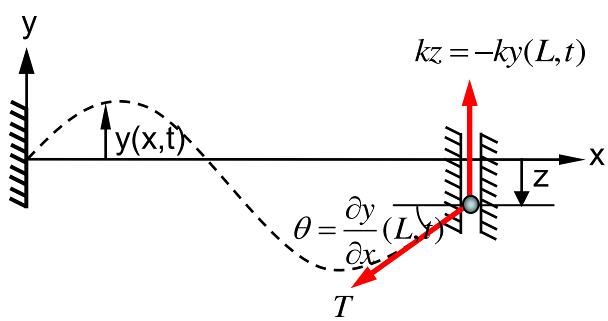
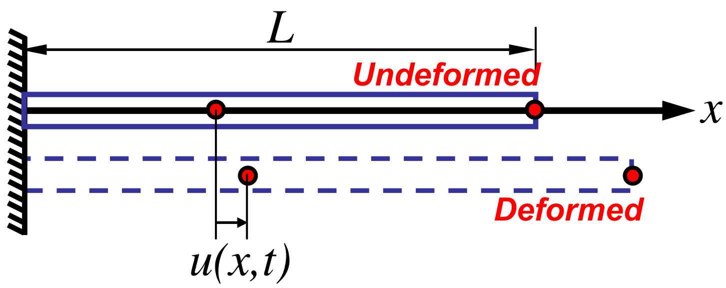


Figure 7. Free-body diagram.

Vibration of Rods

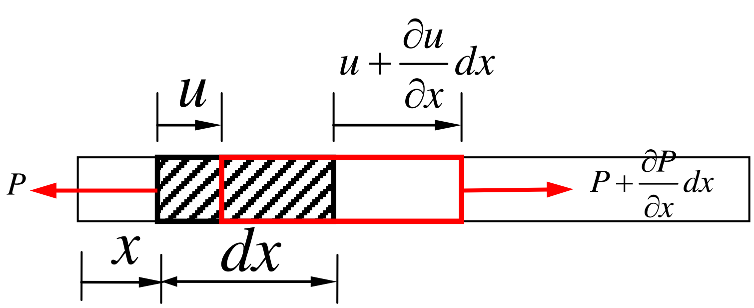
In this section, let us study the free **longitudinal vibration** of rods (bars). Consider a fixed-free rod of length L undergoing longitudinal vibration, as shown below.



The nomenclature adopted in this case is listed below.

ρ	Density (mass per unit volume)
P	Axial force
$u(x,t)$	Longitudinal displacement
A	Cross-sectional area
E	Young's modulus of elasticity

The longitudinal displacement, which is assumed to be small, depends on both the spatial (x) and temporal (t) variables. It is assumed that the displacement is small. In order to study the rod vibration, we need to write down its equation of motion. Consider a rod element, as shown in below.



An infinitesimal element of the rod, shown by the hatched area, undergoes longitudinal motion, and is drawn in its deformed configuration, as shown. The change in length of the element is expressed as:

$$\Delta L = \frac{\partial u}{\partial x} dx \tag{29}$$

The axial strain is then given by:

$$\varepsilon = \frac{\Delta L}{L} = \frac{\partial u}{\partial x} \tag{30}$$

The axial stress in the element can be written as:

$$\sigma = \frac{P}{A} \tag{31}$$

Applying Hooke's law, $\sigma = E\varepsilon$, yields:

$$\frac{P}{A} = E\frac{\partial u}{\partial x} \tag{32}$$

Rearranging:

$$EA\frac{\partial u}{\partial x} = P \tag{33}$$

Differentiating with respect to *x* gives:

$$EA\frac{\partial^2 u}{\partial x^2} = \frac{\partial P}{\partial x} \tag{34}$$

Now apply Newton's second law in the axial direction:

$$\frac{\partial P}{\partial x} dx = \left(\rho A dx\right)\frac{\partial^2 u}{\partial t^2} \tag{35}$$

Combining the previous equations gives:

$$EA\frac{\partial^2 u}{\partial x^2} = \rho A\frac{\partial^2 u}{\partial t^2} \tag{36}$$

which is rearranged to give:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{37}$$

where $c = \sqrt{\dfrac{E}{\rho}}$ is the velocity of wave propagation along the rod. Note the similarity between the wave equation of a string and that of a rod. Once again, the wave equation is a second order partial differential equation that must be satisfied over the entire rod domain, subject to boundary and initial conditions. Also note that the displacement is a function of two independent variables, *x* and *t*.

Solution of the wave equation is emulated by using separation of variables, and the process follows directly from that adopted for strings. Thus we seek a solution in the form:

$$u(x,t) = U(x) \cdot G(t) \tag{38}$$

Upon differentiating partially with respect to t and x yields:

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 U}{dx^2} \cdot G \tag{39}$$

and

$$\frac{\partial^2 u}{\partial t^2} = U \cdot \frac{d^2 G}{dt^2} \tag{40}$$

Substitution into the equation of motion gives:

$$\frac{d^2 U}{dx^2} G = \frac{1}{c^2} U \frac{d^2 G}{dt^2} \tag{41}$$

which is rearranged in the form:

$$\frac{1}{U} \frac{d^2 U}{dx^2} = \frac{1}{c^2} \frac{1}{G} \frac{d^2 G}{dt^2} \tag{42}$$

Once again, we note that the LHS depends only on *x*, whereas the RHS depends only on *t*. In order to satisfy this equation, both sides must be equal to a constant. Let this constant be $-(\omega/c)^2$ for oscillatory motion to prevail. It then follows that:

$$\frac{1}{U} \frac{d^2 U}{dx^2} = -(\omega/c)^2 \tag{43}$$

or:

$$\frac{d^2 U}{dx^2} + (\omega/c)^2 U = 0 \tag{44}$$

and

$$\frac{1}{c^2}\frac{1}{G}\frac{d^2G}{dt^2} = -\left(\omega/c\right)^2 \tag{45}$$

or:

$$\frac{d^2G}{dt^2} + \omega^2 G = 0 \tag{46}$$

These have the general solutions:

$$U(x) = A_1 \sin\left(\omega/c\right)x + A_2 \cos\left(\omega/c\right)x \tag{47}$$

and

$$G(t) = A_3 \sin \omega t + A_4 \cos \omega t \tag{48}$$

The solution $U(x)$ defines the deformation shape, whereas $G(t)$ defines the motion to be harmonic in time. The four constants A_1, A_2, A_3, A_4 are to be determined from the boundary and initial conditions. The natural frequency ω is yet to be determined, and the expression $\left(\omega/c\right)$ is known as the **wave number** or **spatial frequency**. The general solution is finally obtained by:

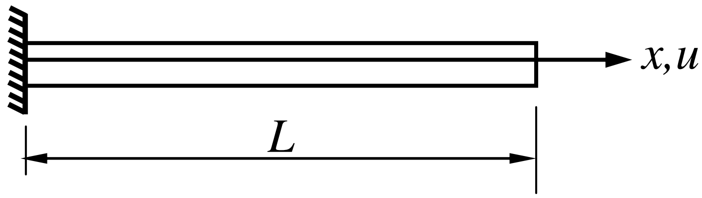
$$u(x,t) = \left(A_1 \sin(\omega/c)x + A_2 \cos(\omega/c)x\right)\cdot\left(A_3 \sin \omega t + A_4 \cos \omega t\right) \tag{49}$$

After some algebraic manipulation, the solution may also be expressed as:

$$\begin{aligned} y(x,t) = a_1 \sin\left((\omega/c)x - \omega t\right) + a_2 \cos\left((\omega/c)x - \omega t\right) + \\ a_3 \sin\left((\omega/c)x + \omega t\right) + a_4 \cos\left((\omega/c)x + \omega t\right) \end{aligned} \tag{50}$$

Example: Fixed-free Rod

As an example, let us investigate the case of a fixed-free rod.



The boundary conditions for this case are:

$u(0,t)=0$

which results in:

$0 = A_2 \cdot G(t)$

from which we get:

$A_2 = 0$

The general solution then becomes:

$u(x,t) = \left(A_1 \sin(\omega/c)x\right) \cdot G(t)$

At the free end $x = L$ the axial force must vanish $P = 0$. But

$$P = EA\frac{\partial u}{\partial x} = EA(\omega/c)\left(A_1 \cos(\omega/c)x\right) \cdot G(t)$$

hence:

$0 = EA(\omega/c)\left(A_1 \cos(\omega L/c)\right) \cdot G(t)$

which implies:

$\cos(\omega L/c) = 0$

which is the **frequency equation** or **characteristic equation** of the system. Solution of this equation is:

$$\omega L/c = \left(\frac{2n-1}{2}\right)\pi \quad , \quad n = 1,2,3,\ldots$$

Hence the natural frequencies of the system are given by:

$$\omega_n = \left(\frac{2n-1}{2}\right)\frac{\pi c}{L} \quad , \quad n = 1,2,3,\ldots$$

and the normal modes of vibration are:

$$U_n(x) = \left(A_{1n} \sin(\omega_n/c)x\right)$$

The solution of each mode becomes:

$$u_n(x,t) = A_{1n} \sin\left((2n-1)\pi x/2L\right) \cdot \left(C_{1n} \sin \omega_n t + D_{1n} \cos \omega_n t\right)$$

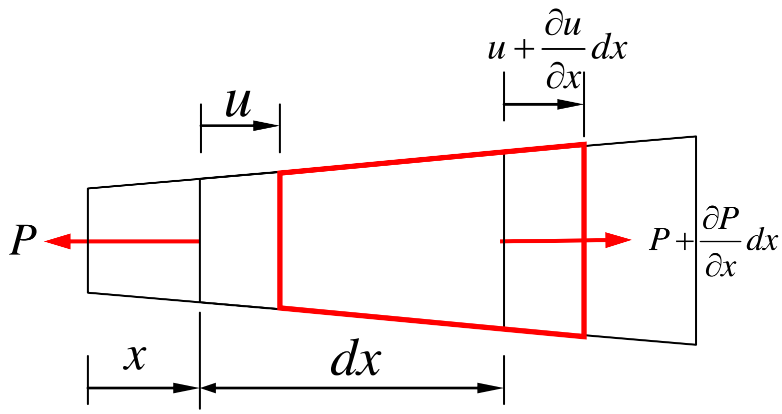
In other words, at each natural frequency, there corresponds a mode shape or an eigenfunction defined by:

$$U_n(x) = A_{1n} \sin\left((2n-1)\pi x/2L\right)$$

and each n represents a normal mode vibration with a natural frequency $\omega_n = \left(\frac{2n-1}{2}\right)\frac{\pi c}{L}$ where A_{1n} are arbitrary constants.

Rod with Non-uniform Cross Section

Consider a rod with a non-uniform cross section, as shown below. The equation of motion can be obtained using the same techniques described previously.



Applying Newton's second law yields:

$$\frac{\partial P}{\partial x} dx = \left(\rho A(x) dx\right) \frac{\partial^2 u}{\partial t^2} \tag{51}$$

which gives:

$$\frac{\partial P}{\partial x} = \left(\rho A(x)\right) \frac{\partial^2 u}{\partial t^2} \tag{52}$$

Note that $A(x)$ describes the variation of cross sectional area along the rod axis. Upon application of Hooke's law, we obtain:

$$P = EA(x) \frac{\partial u}{\partial x} \tag{53}$$

Differentiating with respect to x gives:

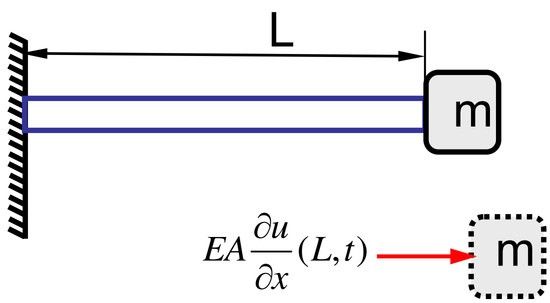
$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\left(EA(x) \frac{\partial u}{\partial x} \right) \tag{54}$$

and hence the equation of motion is expressed as:

$$\rho A(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(EA(x)\frac{\partial u}{\partial x}\right) \qquad (55)$$

Other Boundary Conditions

Finally, let us consider the case where inertial attachments are appended to the rod, as shown below.

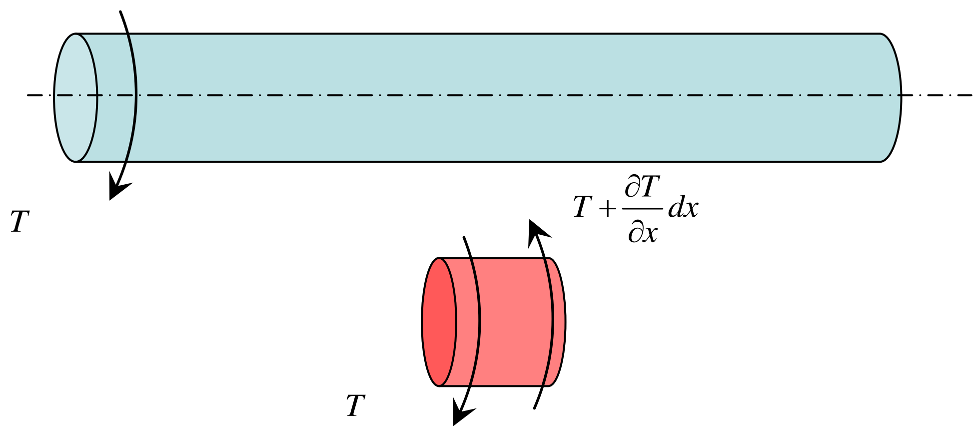


The boundary condition at the fixed end is straight forward. Examination of the boundary condition at the free end leads us to write the equation of motion of the attached mass using Newton's second law. The resulting equation can be expressed as:

$$EA\frac{\partial u}{\partial x}(L,t) = -m\frac{\partial^2 u}{\partial t^2}(L,t) \qquad (56)$$

Torsional Vibration of Rods

Now consider a rod that is subjected to a twisting moment T, as shown below.



The angle of twist can be expressed as:

$$d\theta = \frac{Tdx}{GJ} \tag{57}$$

where

θ : angle of twist

G : modulus of rigidity

J : polar moment of inertia

We can then write:

$$\frac{\partial T}{\partial x} dx = GJ \frac{\partial^2 \theta}{\partial x^2} dx \tag{58}$$

Applying Newton's law to the rod element, we obtain:

$$dxGJ \frac{\partial^2 \theta}{\partial x^2} = \rho J dx \frac{\partial^2 \theta}{\partial t^2} \tag{59}$$

hence

$$\frac{\partial^2 \theta}{\partial t^2} = \frac{G}{J} \frac{\partial^2 \theta}{\partial x^2} \tag{60}$$

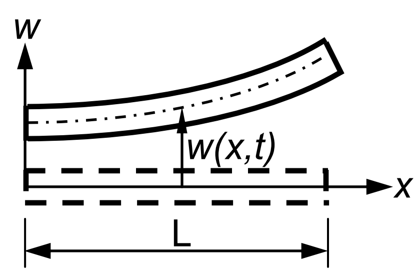
which is in the form:

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2} \tag{61}$$

with $c = \sqrt{\frac{G}{\rho}}$ representing the wave velocity. It is noted that this equation has exactly the same form as equation (37) representing the axial vibration of rods, with the $u \to \theta$, $E \to G$. Thus the evaluating of the natural frequencies, mode shapes and system response follows directly from the equations previously mentioned.

Vibration of Beams

This section deals with the **transverse vibration** of beams. The figure shows an elastic beam drawn in both the undeformed and deformed configurations. Although the figure suggests a cantilever arrangement, the analysis is suited for arbitrary boundary conditions. Transverse displacements, measured from the neutral axis at equilibrium, are designated as *w(x,t)*.



From strength of materials, and adopting the Euler-Bernoulli beam theory, we have:

$$M = -EI\frac{\partial^2 w}{\partial x^2} \tag{62}$$

and:

$$V = \frac{\partial M}{\partial x} \tag{63}$$

where

$E =$ Young's modulus of elasticity [N/m^2]

$I =$ Second moment of area [m^4]

$M =$ bending moment [Nm]

$w =$ Transverse displacement [m]

$V =$ Shear force [N]

Consider an infinitesimal beam element as shown in Fig. 15.

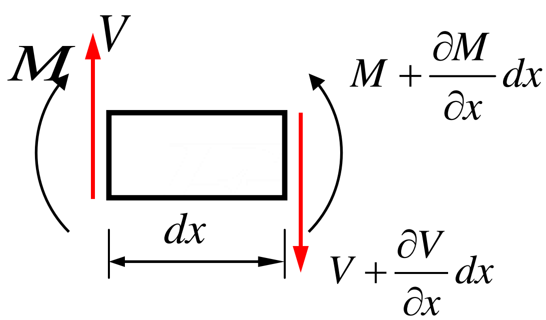


Figure 15. Forces and moments acting on a beam element.

Neglecting rotary inertia, we can apply Newton's law in the vertical (transverse) direction to obtain the equation of motion:

$$\frac{\partial V}{\partial x}dx = \rho A dx \frac{\partial^2 w}{\partial t^2} \tag{64}$$

where A is the cross-sectional area. From equations (57) and (58), we have:

$$V = \frac{\partial M}{\partial x} = \frac{\partial}{\partial x}\left(-EI\frac{\partial^2 w}{\partial x^2}\right) \tag{65}$$

For constant EI we obtain:

$$V = -EI\frac{\partial^3 w}{\partial x^3} \tag{66}$$

or:

$$\frac{\partial V}{\partial x} = -EI\frac{\partial^4 w}{\partial x^4} \tag{67}$$

Combining equation (59) with (62) yields the equation that governs the free transverse vibration of a uniform elastic beam as:

$$-EI\frac{\partial^4 w(x,t)}{\partial x^4} = \rho A\frac{\partial^2 w(x,t)}{\partial t^2} \tag{68}$$

Solution of the above equation can be emulated using the technique of separation of variables, as described in previous sections. In this way, the solution can be expressed as:

$$w(x,t) = W(x)F(t) \tag{69}$$

Substituting (64) into (63) yields:

$$-\frac{EI}{\rho A}\frac{1}{W}\frac{d^4W}{dx^4} = \frac{1}{F}\frac{d^2F}{dt^2} \tag{70}$$

We observe that the left side of (65) depends *only* on x, while the right side depends *only* on t. Because x and t are independent variables, we conclude that both sides of (65) must be equal to a constant. Let this constant be $-\omega^2$. It follows that:

$$\frac{d^2F}{dt^2} + \omega^2 F = 0 \tag{71}$$

which has a solution in the form:

$$F(t) = C_1 \sin \omega t + C_2 \cos \omega t \tag{72}$$

where C_1 and C_2 are constants to be determined from the initial conditions.

Furthermore, we have:

$$\frac{d^4W}{dx^4} - \frac{\rho A}{EI}\omega^2 W = 0 \tag{73}$$

Denoting $\beta^4 = \frac{\rho A}{EI}\omega^2$ we get:

$$\frac{d^4W}{dx^4} - \beta^4 W = 0 \tag{74}$$

The solution of (69) can be shown to be:

$$W(x) = A_1 \sin \beta x + A_2 \cos \beta x + A_3 \sinh \beta x + A_4 \cosh \beta x \tag{75}$$

where A_1, A_2, A_3 and A_4 are constants to be determined from the boundary conditions.

1.12 Example: Cantilever Beam

The boundary conditions for a cantilever beam are given by:

At $x = 0, \quad W = 0, \quad \dfrac{dW}{dx} = 0$

At $x = L, \quad \dfrac{d^2W}{dx^2} = 0, \quad \dfrac{d^3W}{dx^3} = 0$

Upon substitution into (70), and after some algebraic manipulation, we obtain the *eigenfunctions for a cantilever beam* as:

$$W_n(x) = A_n\left[\sin\beta_n x - \sinh\beta_n x - \frac{\sin\beta_n L + \sinh\beta_n L}{\cos\beta_n L + \cosh\beta_n L}\left(\cos\beta_n x - \cosh\beta_n x\right)\right] \tag{76}$$

where β_n is obtained by solving

$$\cos\beta L \cosh\beta L = -1 \tag{77}$$

Equation (72) can be solved numerically to give the *eigenvalues* $\beta_1 L, \beta_2 L, \ldots, \beta_n L$. The first three solutions can be shown to be:

$$\begin{aligned} \beta_1 &= 1.8751/L \\ \beta_2 &= 4.6941/L \\ \beta_3 &= 7.8548/L \end{aligned}$$

Now the natural frequencies can be obtained from $\omega_n = \beta_n^2\sqrt{\dfrac{EI}{\rho A}}$ and the first three values are obtained as:

$$\begin{aligned} \omega_1 &= (1.8751)^2\sqrt{\frac{EI}{\rho A L^4}} \\ \omega_2 &= (4.6941)^2\sqrt{\frac{EI}{\rho A L^4}} \\ \omega_3 &= (7.8548)^2\sqrt{\frac{EI}{\rho A L^4}} \end{aligned}$$

Figure 16 shows a typical frequency response plot of a cantilever beam, acted upon by a harmonic force at its tip. The amplitude of the tip motion is plotted as a function of the excitation frequency. The peaks occur at the *natural frequencies* of the system, and the deformation pattern of the beam (eigenfunctions) at each frequency is plotted at the top.

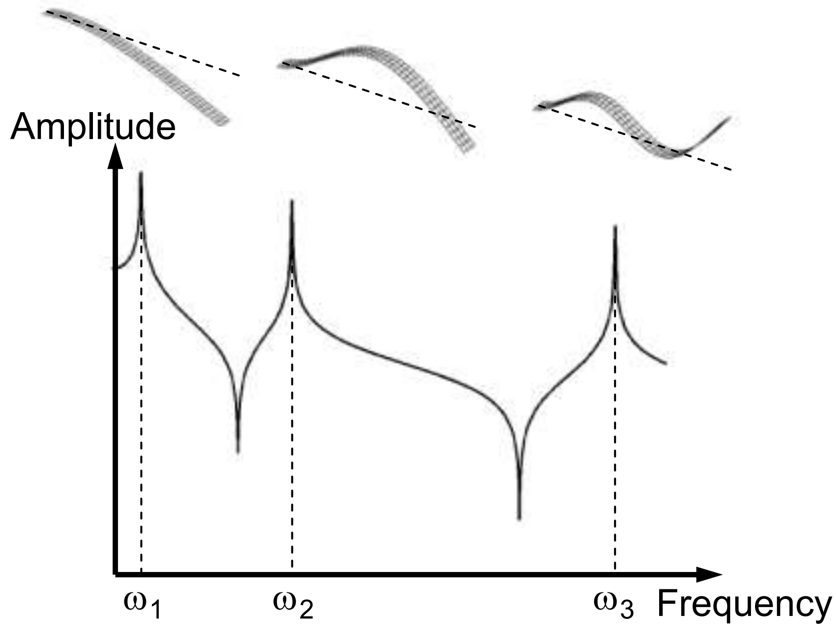


Figure 16. Frequency response of a beam.

VIBRATION OF CONTINUOUS SYSTEMS

Introduction

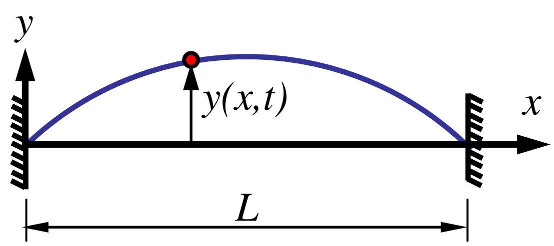
Models of vibratory systems can be divided into two broad classes, *lumped* and *continuous*, depending on the nature of the parameters. In the case of lumped systems, the components are discrete, with the mass assumed to be rigid and concentrated at individual points, and with the stiffness taking the form of massless springs connecting the rigid masses. The masses and springs represent the system parameters, and we refer to such models as *discrete* or *lumped-parameter* models. The motion of discrete systems is governed by *ordinary* differential equations. *Continuous* systems, on the other hand, differ from discrete systems in that the mass and elasticity are continuously distributed. Such systems are also known as *distributed-parameter* systems, and examples include strings, rods, beams, plates and shells. While discrete systems possess a finite number of degrees of freedom, continuous systems have an infinite number of degrees of freedom because we need an infinite number of coordinates to specify the displacement of every point in an elastic body. The displacement in this case depends on two independent variables, namely x and t. As a result, the motion of continuous systems is governed by partial differential equations to be satisfied over the entire domain of the system, subject to boundary conditions and initial conditions.

Although discrete systems and continuous system may appear entirely different in nature, the difference is more in form than concept. As a matter of fact, a certain physical system can be modeled either as discrete or as distributed, depending on the objectives of the analysis. It turns out that discrete and continuous systems are indeed closely connected, and thus it comes as no surprise that both systems possess natural frequencies and normal modes of vibration.

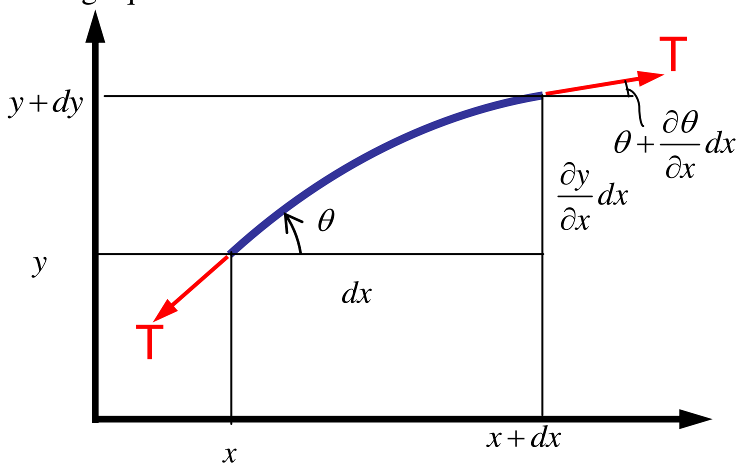
In this topic we will study the free and forced vibration of continuous systems. Emphasis will be placed on studying the vibration of taught strings, rods and beams. This covers a broad class of engineering applications, as many practical systems can be modeled by one or more of such elements in order to study the dynamic behavior

Vibration of Strings

The figure shows a fixed-fixed string of length L. The string is initially under tension T and the aim is to study the **transverse vibrations** denoted by the displacement $y(x,t)$, measured from the equilibrium position. It is assumed that both displacement and slope are small.



It is also assumed that the tension force in the string remains constant during vibration, which follows from the previous assumptions of small displacements. As in all continuous systems, the displacement variable depends on both the spatial (x) and temporal (t) coordinates. A free body diagram of a string element is shown below. Neglecting gravity effects, we can apply Newton's second law on the string element to obtain the governing equation of motion.



Applying $\uparrow \Sigma F_y = m \cdot a_y$ gives:

$$T\sin\left(\theta+\frac{\partial\theta}{\partial x}dx\right)-T\sin\theta=\rho dx\frac{\partial^2 y}{\partial t^2} \tag{1}$$

where ρ is the **mass per unit length** of the string. For small displacements, $\sin\theta \simeq \theta$, hence we obtain:

$$T\frac{\partial\theta}{\partial x}=\rho\frac{\partial^2 y}{\partial t^2} \tag{2}$$

But $\theta = \dfrac{\partial y}{\partial x}$ hence:

$$T\frac{\partial^2 y}{\partial x^2} = \rho\frac{\partial^2 y}{\partial t^2} \tag{3}$$

which can be written as:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \tag{4}$$

which is known as the one-dimensional wave equation and $c = \sqrt{\dfrac{T}{\rho}}$ is the velocity of wave propagation along the string. The wave equation is a partial differential equation, and the same form will be encountered in similar problems involving the dynamics of distributed-parameter models. The equation must be satisfied over the entire domain and is subject to **boundary conditions** as well as **initial conditions**. Accordingly, the problem posed is both a *boundary value problem (BVP)* and an *initial value problem (IVP)* from a mathematical point pf view.

We now seek the solution of the wave equation, which represents the variation of the transverse displacement at any point along the string and at any time for an arbitrary string that is set in motion by certain initial conditions and left to vibrate freely. This solution is emulated by the using the principle of separation of variables. In this way, the transverse displacement can be expressed as:

$$y(x,t) = Y(x)\cdot G(t) \tag{5}$$

It follows that:

$$\frac{\partial^2 y}{\partial x^2} = \frac{d^2Y}{dx^2}\cdot G \tag{6}$$

and

$$\frac{\partial^2 y}{\partial t^2} = Y\cdot\frac{d^2G}{dt^2} \tag{7}$$

Substitution into the equation of motion (4) yields:

$$\frac{d^2Y}{dx^2}G = \frac{1}{c^2}Y\frac{d^2G}{dt^2} \tag{8}$$

which can also be written as:

$$\frac{1}{Y}\frac{d^2Y}{dx^2} = \frac{1}{c^2}\frac{1}{G}\frac{d^2G}{dt^2} \tag{9}$$

It is noted that the left-hand-side (LHS) of the above equation depends **only** on the spatial variable x, whereas the RHS depends only on the temporal variable, t. In order to satisfy the equation, both sides of equation (9) must be equal to a constant. Let this constant be $-(\omega/c)^2$. A negative constant was conveniently selected because this choice leads to an oscillatory motion. The choice of a zero or positive constant does not yield a vibratory motion, and therefore must be excluded. For example, if a zero constant was chosen, this leads to:

$$\frac{1}{c^2}\frac{1}{G}\frac{d^2G}{dt^2}=0$$

or

$$\frac{d^2G}{dt^2}=0$$

whose solution is given by:

$$G=c_1t+c_2$$

which is rejected because it indicates a solution that increases linearly with time. It can be shown that the choice of a positive constant gives rise to two terms; one exponentially increasing and the other exponentially decreasing.

Adopting the negative constant choice, and substituting into the equation of motion gives:

$$\frac{1}{Y}\frac{d^2Y}{dx^2}=-(\omega/c)^2 \tag{10}$$

which can be written as:

$$\frac{d^2Y}{dx^2}+(\omega/c)^2\,Y=0 \tag{11}$$

Furthermore,

$$\frac{1}{c^2}\frac{1}{G}\frac{d^2G}{dt^2}=-(\omega/c)^2 \tag{12}$$

which can similarly be expressed as:

$$\frac{d^2G}{dt^2}+\omega^2G=0 \tag{13}$$

These have the general solutions:

$$Y(x)=A\sin(\omega/c)\,x+B\cos(\omega/c)\,x \tag{14}$$

And

$$G(t)=C\sin\omega t+D\cos\omega t \tag{15}$$

The 4 constants A, B, C and D are to be determined from the boundary conditions (BC's) and initial conditions (IC's). It also worthy to note that equation (14) defines the deformation shape, whereas equation (15) defines the motion to be harmonic in time. It becomes appropriate now to define the unknown constant ω as the natural frequency of the system, and $\left(\omega/c\right)$ as the wave number or spatial frequency. The general solution may then be expressed as:

$$y(x,t) = \left(A\sin(\omega/c)x + B\cos(\omega/c)x\right)\cdot\left(C\sin\omega t + D\cos\omega t\right) \tag{16}$$

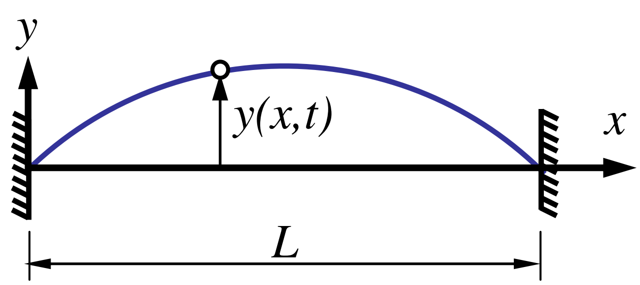
Alternatively, and after some algebraic manipulation, the above solution may also be written as:

$$\begin{aligned} y(x,t) = a_1\sin\left((\omega/c)x - \omega t\right) + a_2\cos\left((\omega/c)x - \omega t\right) + \\ a_3\sin\left((\omega/c)x + \omega t\right) + a_4\cos\left((\omega/c)x + \omega t\right) \end{aligned} \tag{17}$$

Once again, the solution must contain 4 unknown constants.

Example: Fixed-fixed string

Let us now consider the case of a string that is fixed at both ends, as shown.



The imposed boundary conditions indicate that the string displacement at both ends must be equal to zero, or:

$y(0,t) = 0$ and $y(L,t) = 0$. The general solution is:

$$\begin{aligned} y(x,t) &= Y(x)\cdot G(t) \\ &= \left(A\sin(\omega/c)x + B\cos(\omega/c)x\right)\cdot\left(C\sin\omega t + D\cos\omega t\right) \end{aligned}$$

Substitution of the first BC into the general solution gives:

$$0 = B\cdot\left(C\sin\omega t + D\cos\omega t\right)$$

which implies $B = 0$. The general solution hence becomes:

$y(x,t) = \left(A\sin(\omega/c)x\right)\cdot\left(C\sin\omega t + D\cos\omega t\right)$

Substitution of the second BC into the solution gives:

$0 = \left(A\sin(\omega L/c)\right)\cdot\left(C\sin\omega t + D\cos\omega t\right)$

which implies:

$\sin(\omega L/c) = 0$

hence

$\omega L/c = n\pi \quad , \quad n = 1, 2, 3, \ldots$

The above equation is termed the **frequency equation** or **characteristic equation** of the system, as it gives values of the system natural frequencies. Clearly, the system possesses an *infinite number of natural frequencies*, as suggested earlier. Having obtained the natural frequencies, the solution at any frequency or mode is expressed by:

$$\begin{aligned} y_n(x,t) &= \left(A_n\sin(n\pi x/L)\right)\left(C_n\sin\omega_n t + D_n\cos\omega_n t\right) \\ &= Y_n(x)\cdot G_n(t) \end{aligned}$$

Therefore, at each natural frequency, there corresponds a certain **mode shape** or an **eigenfunction** defined by

$Y_n(x) = A_n\sin\left(n\pi x/L\right)$

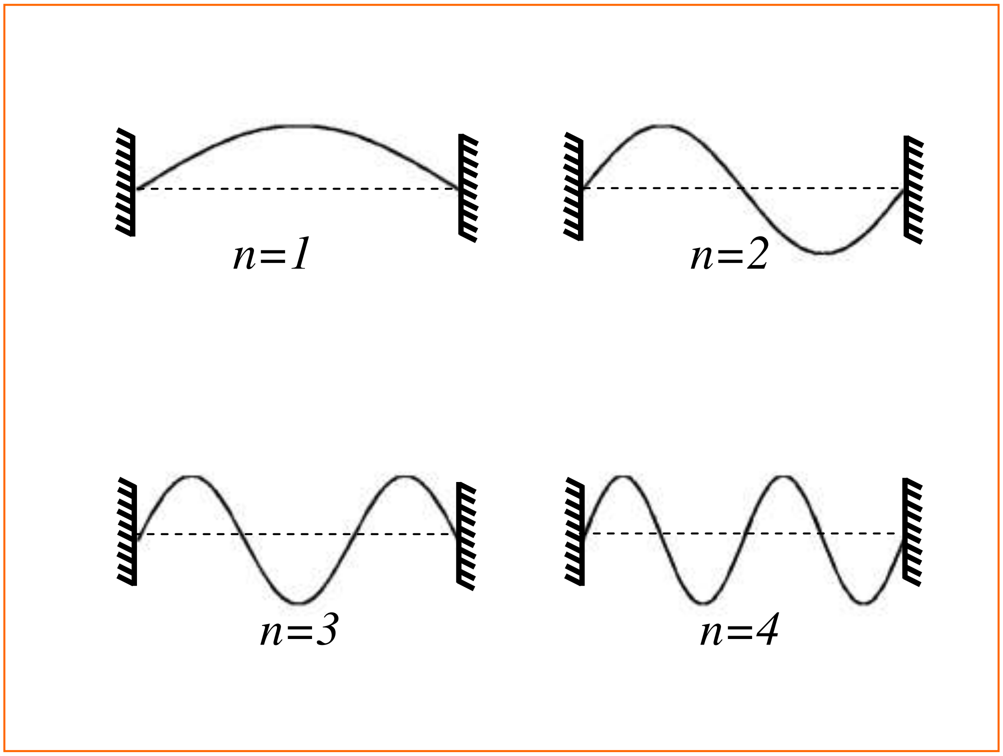
where each "*n*" represents a normal mode vibration with a natural frequency

$$\omega_n = \frac{n\pi c}{L}$$

and mode shape

$Y_n(x) = A_n\sin\left(n\pi x/L\right)$

where A_n are arbitrary constants. The figure below shows the first few modes of the string, as obtained from the above analysis.



The general solution is given by:

$$y_n(x,t) = \left(A_n \sin(n\pi x/L)\right)\left(C_n \sin\omega_n t + D_n \cos\omega_n t\right)$$

where the terms in the first bracket define the displacement pattern at mode "*n*", and the terms on the last bracket define a harmonic motion at the corresponding natural frequency. The free vibration solution is finally obtained as the sum of all modes of vibration, or:

$$y(x,t) = \sum_{n=1}^{\infty}\left(C_n \sin\omega_n t + D_n \cos\omega_n t\right)\sin(n\pi x/L)$$

where C_n, D_n are constants to be determined from the IC's.

The eigenfunctions can also be shown to possess an orthogonality property (see next section) which is given by:

$$\int_0^L Y_n(x)Y_m(x)dx = \begin{cases} 0 & n \neq m \\ h_n & n = m \end{cases}$$

For a fixed-fixed string, this becomes:

$$\int_0^L \sin(n\pi x/L)\sin(m\pi x/L)dx = \begin{cases} 0 & n \neq m \\ L/2 & n = m \end{cases}$$

In order to study the complete free vibration problem, the initial conditions must also be defined. Assume that the string is subjected to the following initial conditions:

$$y(x,0) = f(x) \quad , \quad \dot{y}(x,0) = g(x)$$

Substitution into the general solution gives:

$$f(x) = \sum_{n=1}^{\infty} D_n \sin(n\pi x/L)$$

and

$$g(x) = \sum_{n=1}^{\infty} C_n \omega_n \sin(n\pi x/L)$$

In order to obtain the constants C_n, D_n we multiply the above equations by $\sin(n\pi x/L)$ and integrate over the string length. By using the orthogonality codition, we get:

$$D_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)dx$$

and

$$C_n = \frac{2}{L\omega_n}\int_0^L g(x)\sin(n\pi x/L)dx$$

It thus follows that the constants C_n, D_n determine the **contribution** of each mode to the general solution. Now consider the special case where the initial conditions impose a displacement pattern that **coincides** with one of the natural modes, say mode "*k*", and that the initial velocity is zero. In this way, we have:

$$f(x) = \sin(k\pi x/L)$$

and

$$g(x) = 0$$

It follows that:

$$D_n = \begin{cases} 0 & k \neq n \\ 1 & k = n \end{cases}$$

and

$$C_n = 0$$

Therefore, the general solution reduces to:

$$y(x,t) = \sin(k\pi x/L)\cos\omega_k t$$

That is, the string vibrates in its "k^{th}" mode due to the fact that the initial conditions cause only the "k^{th}" mode to be excited. The continuous system in this special case behaves like a single DOF system. The same notion can be observed for a multi-DOF system. If, on the other hand, an impulse is given to the system, it can be shown that a wide spectrum of frequencies or modes are excited, and hence the system response will contain a summation of a large number of modes. In practice, the contribution of the higher modes is usually smaller, and the system response will be almost predominantly be described by the fundamental (i.e. first) mode, together with only a few higher modes.

Orthogonality of Eigenfunctions

Let us now prove the orthogonality condition for a fixed-fixed string. The same technique can be applied to other forms of boundary conditions. Recall Eq. (11) obtained previously:

$$\frac{d^2Y}{dx^2}+\left(\omega/c\right)^2 Y=0 \tag{18}$$

This equation must be satisfied at all natural frequencies or all normal modes of vibration. Consequently, at an arbitrary mode "*m*", we have ω_m, Y_m as the natural frequency and associated eigenfuction, respectively. Similarly, at mode "*n*", we have ω_n, Y_n. Let us first assume that these two modes are distinct. It follows from Eq. (18) that:

$$-\frac{d^2Y_m}{dx^2}=\left(\omega_m/c\right)^2 Y_m=0 \tag{19}$$

and

$$-\frac{d^2Y_n}{dx^2}=\left(\omega_n/c\right)^2 Y_n=0 \tag{20}$$

Multiplying Eq. (19) by Y_n and integrating over the string length yields:

$$-\int_0^L Y_n\frac{d^2Y_m}{dx^2}dx=\int_0^L\left(\omega_m/c\right)^2 Y_mY_ndx \tag{21}$$

The LHS of the above equation can be integrated by parts:

$$LHS=-\left[Y_n\frac{dY_m}{dx}\bigg|_0^L-\int_0^L\frac{dY_m}{dx}\frac{dY_n}{dx}dx\right]=\int_0^L\frac{dY_m}{dx}\frac{dY_n}{dx}dx \tag{22}$$

and the first term vanishes due to the boundary conditions defined (both ends fixed). It follows that:

$$\int_0^L\frac{dY_m}{dx}\frac{dY_n}{dx}dx=\int_0^L\left(\omega_m/c\right)^2 Y_mY_ndx \tag{23}$$

Similarly, we can multiply equation (20) by Y_m and integrate over the string length. This yields:

$$\int_0^L\frac{dY_n}{dx}\frac{dY_m}{dx}dx=\int_0^L\left(\omega_n/c\right)^2 Y_mY_ndx \tag{24}$$

It is noted that the LHS of Eqs. (23) and (24) are identical, irrespective of the order of multiplication. Subtracting Eq. (24) from Eq. (23) gives:

$$0 = \frac{\omega_m^2 - \omega_n^2}{c^2} \int_0^L Y_m Y_n dx \tag{25}$$

But ω_m, ω_n are two distinct modes, i.e. $\omega_m \neq \omega_n$, therefore:

$$\int_0^L Y_m Y_n dx = 0 \quad , \quad m \neq n \tag{26}$$

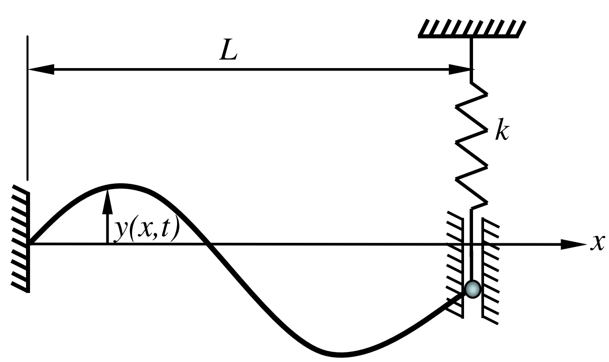
which proves the orthogonality condition for eigenfunctions of a fixed-fixed string. This condition also holds true for other types of BC's. Moreover, since eigenfunctions can be arbitrarily scaled (or *normalized*), we can write:

$$\int_0^L Y_m Y_m dx = h_m \quad , \quad m = n \tag{27}$$

where h_m is a constant.

Elastic or Inertial Attachments

Finally, let us now consider other forms of boundary conditions. Figure 6 shows a string that is fixed at one end and attached to a spring at the other. The boundary condition at the fixed end $x = 0$ is given by $y(0,t) = 0$. This is called a **geometric boundary condition**, because it describes a *specified* displacement. Such a condition is also known as an **essential** or **imposed** boundary condition.



Elastic attachments.

The boundary condition at the other end is not so obvious at first sight. Indeed it becomes appropriate to draw a free-body diagram of the string in order to investigate the force interaction. Such a free-body diagram is shown in Fig.7. At $x = L$ we need to balance forces in the vertical direction. Thus we have:

$$T\frac{\partial y}{\partial x}(L,t) = -ky(L,t) \tag{28}$$

This is called a **natural boundary condition** (also known as **dynamic** or **additional** boundary condition) as it describes forces and moments acting on the system. We can then proceed with the solution in the same way described above in order to obtain the natural frequencies, eigenfunctions and response to initial conditions.

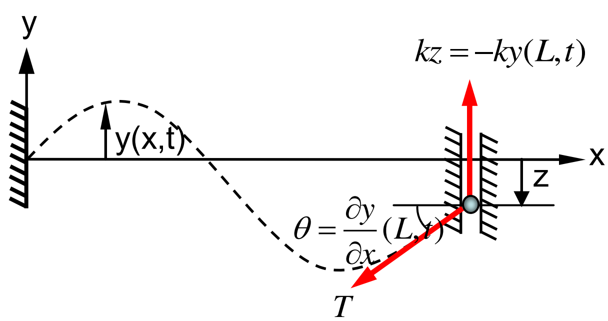
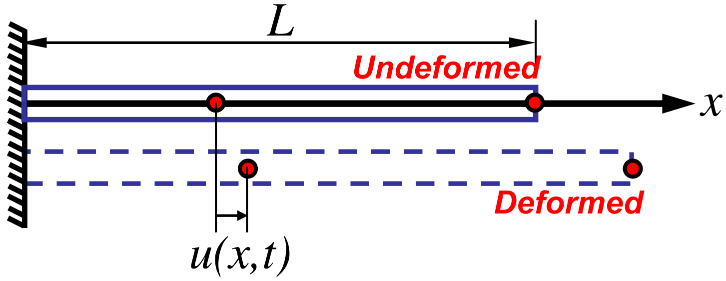


Figure 7. Free-body diagram.

Vibration of Rods

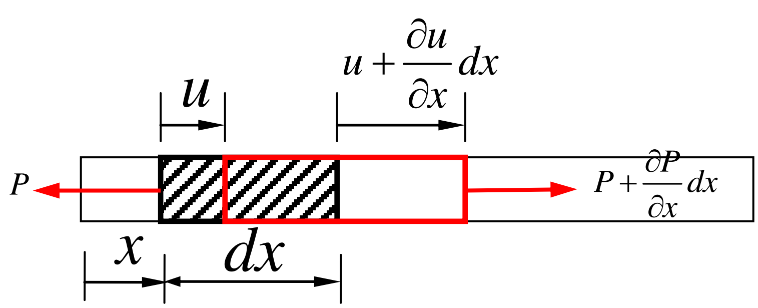
In this section, let us study the free **longitudinal vibration** of rods (bars). Consider a fixed-free rod of length L undergoing longitudinal vibration, as shown below.



The nomenclature adopted in this case is listed below.

ρ	Density (mass per unit volume)
P	Axial force
$u(x,t)$	Longitudinal displacement
A	Cross-sectional area
E	Young's modulus of elasticity

The longitudinal displacement, which is assumed to be small, depends on both the spatial (x) and temporal (t) variables. It is assumed that the displacement is small. In order to study the rod vibration, we need to write down its equation of motion. Consider a rod element, as shown in below.



An infinitesimal element of the rod, shown by the hatched area, undergoes longitudinal motion, and is drawn in its deformed configuration, as shown. The change in length of the element is expressed as:

$$\Delta L = \frac{\partial u}{\partial x} dx \tag{29}$$

The axial strain is then given by:

$$\varepsilon = \frac{\Delta L}{L} = \frac{\partial u}{\partial x} \tag{30}$$

The axial stress in the element can be written as:

$$\sigma = \frac{P}{A} \tag{31}$$

Applying Hooke's law, $\sigma = E\varepsilon$, yields:

$$\frac{P}{A} = E\frac{\partial u}{\partial x} \tag{32}$$

Rearranging:

$$EA\frac{\partial u}{\partial x} = P \tag{33}$$

Differentiating with respect to *x* gives:

$$EA\frac{\partial^2 u}{\partial x^2} = \frac{\partial P}{\partial x} \tag{34}$$

Now apply Newton's second law in the axial direction:

$$\frac{\partial P}{\partial x} dx = \left(\rho A dx\right)\frac{\partial^2 u}{\partial t^2} \tag{35}$$

Combining the previous equations gives:

$$EA\frac{\partial^2 u}{\partial x^2} = \rho A\frac{\partial^2 u}{\partial t^2} \tag{36}$$

which is rearranged to give:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{37}$$

where $c = \sqrt{\frac{E}{\rho}}$ is the velocity of wave propagation along the rod. Note the similarity between the wave equation of a string and that of a rod. Once again, the wave equation is a second order partial differential equation that must be satisfied over the entire rod domain, subject to boundary and initial conditions. Also note that the displacement is a function of two independent variables, *x* and *t*.

Solution of the wave equation is emulated by using separation of variables, and the process follows directly from that adopted for strings. Thus we seek a solution in the form:

$$u(x,t) = U(x) \cdot G(t) \tag{38}$$

Upon differentiating partially with respect to t and x yields:

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 U}{dx^2} \cdot G \tag{39}$$

and

$$\frac{\partial^2 u}{\partial t^2} = U \cdot \frac{d^2 G}{dt^2} \tag{40}$$

Substitution into the equation of motion gives:

$$\frac{d^2 U}{dx^2} G = \frac{1}{c^2} U \frac{d^2 G}{dt^2} \tag{41}$$

which is rearranged in the form:

$$\frac{1}{U} \frac{d^2 U}{dx^2} = \frac{1}{c^2} \frac{1}{G} \frac{d^2 G}{dt^2} \tag{42}$$

Once again, we note that the LHS depends only on *x*, whereas the RHS depends only on *t*. In order to satisfy this equation, both sides must be equal to a constant. Let this constant be $-\left(\omega/c\right)^2$ for oscillatory motion to prevail. It then follows that:

$$\frac{1}{U} \frac{d^2 U}{dx^2} = -\left(\omega/c\right)^2 \tag{43}$$

or:

$$\frac{d^2 U}{dx^2} + \left(\omega/c\right)^2 U = 0 \tag{44}$$

and

$$\frac{1}{c^2}\frac{1}{G}\frac{d^2G}{dt^2} = -\left(\omega/c\right)^2 \tag{45}$$

or:

$$\frac{d^2G}{dt^2} + \omega^2 G = 0 \tag{46}$$

These have the general solutions:

$$U(x) = A_1 \sin\left(\omega/c\right)x + A_2 \cos\left(\omega/c\right)x \tag{47}$$

and

$$G(t) = A_3 \sin\omega t + A_4 \cos\omega t \tag{48}$$

The solution $U(x)$ defines the deformation shape, whereas $G(t)$ defines the motion to be harmonic in time. The four constants A_1, A_2, A_3, A_4 are to be determined from the boundary and initial conditions. The natural frequency ω is yet to be determined, and the expression $\left(\omega/c\right)$ is known as the **wave number** or **spatial frequency**. The general solution is finally obtained by:

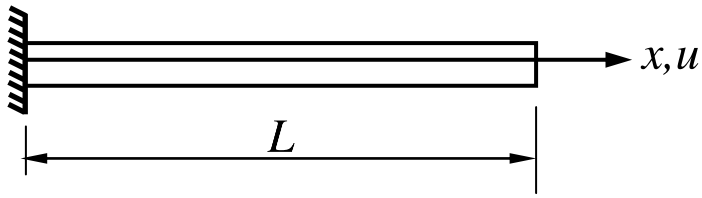
$$u(x,t) = \left(A_1 \sin(\omega/c)x + A_2 \cos(\omega/c)x\right)\cdot\left(A_3 \sin\omega t + A_4 \cos\omega t\right) \tag{49}$$

After some algebraic manipulation, the solution may also be expressed as:

$$\begin{aligned} y(x,t) = a_1 \sin\left((\omega/c)x - \omega t\right) + a_2 \cos\left((\omega/c)x - \omega t\right) + \\ a_3 \sin\left((\omega/c)x + \omega t\right) + a_4 \cos\left((\omega/c)x + \omega t\right) \end{aligned} \tag{50}$$

Example: Fixed-free Rod

As an example, let us investigate the case of a fixed-free rod.



The boundary conditions for this case are:

$u(0,t) = 0$

which results in:

$0 = A_2 \cdot G(t)$

from which we get:

$A_2 = 0$

The general solution then becomes:

$u(x,t) = \left(A_1 \sin(\omega/c)x\right) \cdot G(t)$

At the free end $x = L$ the axial force must vanish $P = 0$. But

$$P = EA\frac{\partial u}{\partial x} = EA(\omega/c)\left(A_1 \cos(\omega/c)x\right) \cdot G(t)$$

hence:

$0 = EA(\omega/c)\left(A_1 \cos(\omega L/c)\right) \cdot G(t)$

which implies:

$\cos(\omega L/c) = 0$

which is the **frequency equation** or **characteristic equation** of the system. Solution of this equation is:

$$\omega L/c = \left(\frac{2n-1}{2}\right)\pi \quad , \quad n = 1, 2, 3, \ldots$$

Hence the natural frequencies of the system are given by:

$$\omega_n = \left(\frac{2n-1}{2}\right)\frac{\pi c}{L} \quad , \quad n = 1, 2, 3, \ldots$$

and the normal modes of vibration are:

$$U_n(x) = \left(A_{1n} \sin(\omega_n / c) x\right)$$

The solution of each mode becomes:

$$u_n(x,t) = A_{1n} \sin\left((2n-1)\pi x / 2L\right) \cdot \left(C_{1n} \sin \omega_n t + D_{1n} \cos \omega_n t\right)$$

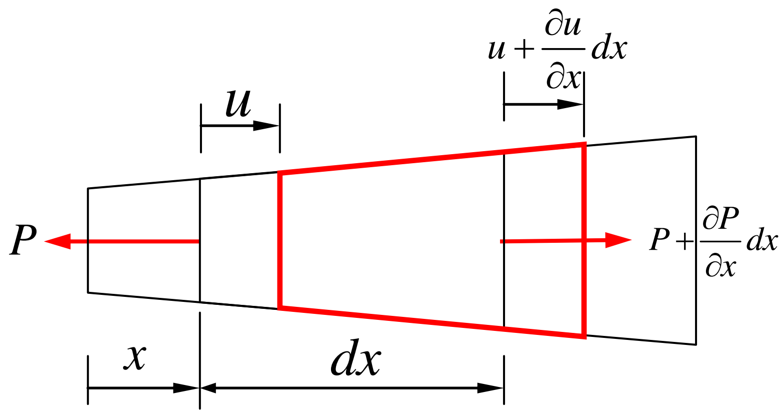
In other words, at each natural frequency, there corresponds a mode shape or an eigenfunction defined by:

$$U_n(x) = A_{1n} \sin\left((2n-1)\pi x / 2L\right)$$

and each n represents a normal mode vibration with a natural frequency $\omega_n = \left(\frac{2n-1}{2}\right)\frac{\pi c}{L}$ where A_{1n} are arbitrary constants.

Rod with Non-uniform Cross Section

Consider a rod with a non-uniform cross section, as shown below. The equation of motion can be obtained using the same techniques described previously.



Applying Newton's second law yields:

$$\frac{\partial P}{\partial x}dx=\left(\rho A(x)dx\right)\frac{\partial^2 u}{\partial t^2} \tag{51}$$

which gives:

$$\frac{\partial P}{\partial x}=\left(\rho A(x)\right)\frac{\partial^2 u}{\partial t^2} \tag{52}$$

Note that $A(x)$ describes the variation of cross sectional area along the rod axis. Upon application of Hooke's law, we obtain:

$$P=EA(x)\frac{\partial u}{\partial x} \tag{53}$$

Differentiating with respect to x gives:

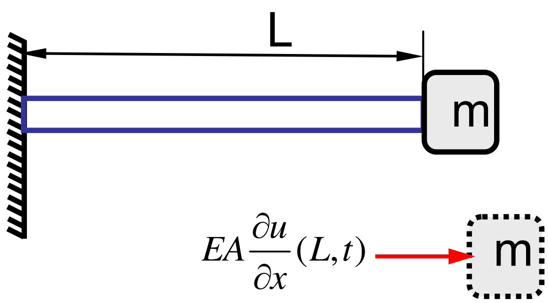
$$\frac{\partial P}{\partial x}=\frac{\partial}{\partial x}\left(EA(x)\frac{\partial u}{\partial x}\right) \tag{54}$$

and hence the equation of motion is expressed as:

$$\rho A(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(EA(x)\frac{\partial u}{\partial x}\right) \tag{55}$$

Other Boundary Conditions

Finally, let us consider the case where inertial attachments are appended to the rod, as shown below.

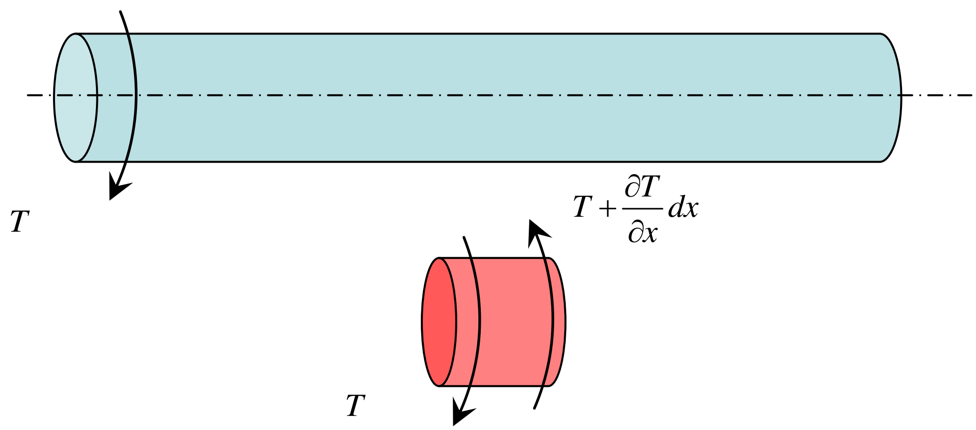


The boundary condition at the fixed end is straight forward. Examination of the boundary condition at the free end leads us to write the equation of motion of the attached mass using Newton's second law. The resulting equation can be expressed as:

$$EA\frac{\partial u}{\partial x}(L,t) = -m\frac{\partial^2 u}{\partial t^2}(L,t) \tag{56}$$

Torsional Vibration of Rods

Now consider a rod that is subjected to a twisting moment T, as shown below.



The angle of twist can be expressed as:

$$d\theta = \frac{Tdx}{GJ} \tag{57}$$

where

θ : angle of twist

G : modulus of rigidity

J : polar moment of inertia

We can then write:

$$\frac{\partial T}{\partial x} dx = GJ \frac{\partial^2 \theta}{\partial x^2} dx \tag{58}$$

Applying Newton’s law to the rod element, we obtain:

$$dxGJ \frac{\partial^2 \theta}{\partial x^2} = \rho J dx \frac{\partial^2 \theta}{\partial t^2} \tag{59}$$

hence

$$\frac{\partial^2 \theta}{\partial t^2} = \frac{G}{J} \frac{\partial^2 \theta}{\partial x^2} \tag{60}$$

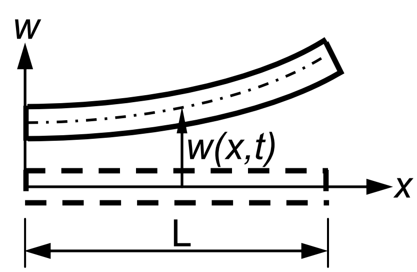
which is in the form:

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \theta}{\partial t^2} \tag{61}$$

with $c = \sqrt{\frac{G}{\rho}}$ representing the wave velocity. It is noted that this equation has exactly the same form as equation (37) representing the axial vibration of rods, with the $u \to \theta$, $E \to G$. Thus the evaluating of the natural frequencies, mode shapes and system response follows directly from the equations previously mentioned.

Vibration of Beams

This section deals with the **transverse vibration** of beams. The figure shows an elastic beam drawn in both the undeformed and deformed configurations. Although the figure suggests a cantilever arrangement, the analysis is suited for arbitrary boundary conditions. Transverse displacements, measured from the neutral axis at equilibrium, are designated as *w(x,t)*.



From strength of materials, and adopting the Euler-Bernoulli beam theory, we have:

$$M = -EI\frac{\partial^2 w}{\partial x^2} \tag{62}$$

and:

$$V = \frac{\partial M}{\partial x} \tag{63}$$

where

$E =$ Young's modulus of elasticity [N/m^2]

$I =$ Second moment of area [m^4]

$M =$ bending moment [Nm]

$w =$ Transverse displacement [m]

$V =$ Shear force [N]

Consider an infinitesimal beam element as shown in Fig. 15.

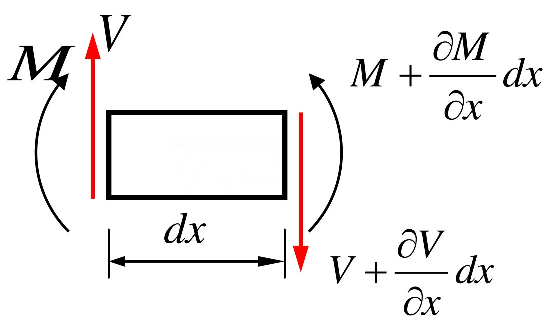


Figure 15. Forces and moments acting on a beam element.

Neglecting rotary inertia, we can apply Newton's law in the vertical (transverse) direction to obtain the equation of motion:

$$\frac{\partial V}{\partial x} dx = \rho A dx \frac{\partial^2 w}{\partial t^2} \tag{64}$$

where A is the cross-sectional area. From equations (57) and (58), we have:

$$V = \frac{\partial M}{\partial x} = \frac{\partial}{\partial x}\left(-EI \frac{\partial^2 w}{\partial x^2}\right) \tag{65}$$

For constant EI we obtain:

$$V = -EI \frac{\partial^3 w}{\partial x^3} \tag{66}$$

or:

$$\frac{\partial V}{\partial x} = -EI \frac{\partial^4 w}{\partial x^4} \tag{67}$$

Combining equation (59) with (62) yields the equation that governs the free transverse vibration of a uniform elastic beam as:

$$-EI\frac{\partial^4 w(x,t)}{\partial x^4} = \rho A \frac{\partial^2 w(x,t)}{\partial t^2} \tag{68}$$

Solution of the above equation can be emulated using the technique of separation of variables, as described in previous sections. In this way, the solution can be expressed as:

$$w(x,t) = W(x)F(t) \tag{69}$$

Substituting (64) into (63) yields:

$$-\frac{EI}{\rho A}\frac{1}{W}\frac{d^4 W}{dx^4} = \frac{1}{F}\frac{d^2 F}{dt^2} \tag{70}$$

We observe that the left side of (65) depends *only* on x, while the right side depends *only* on t. Because x and t are independent variables, we conclude that both sides of (65) must be equal to a constant. Let this constant be $-\omega^2$. It follows that:

$$\frac{d^2 F}{dt^2} + \omega^2 F = 0 \tag{71}$$

which has a solution in the form:

$$F(t) = C_1 \sin \omega t + C_2 \cos \omega t \tag{72}$$

where C_1 and C_2 are constants to be determined from the initial conditions.

Furthermore, we have:

$$\frac{d^4 W}{dx^4} - \frac{\rho A}{EI}\omega^2 W = 0 \tag{73}$$

Denoting $\beta^4 = \frac{\rho A}{EI}\omega^2$ we get:

$$\frac{d^4 W}{dx^4} - \beta^4 W = 0 \tag{74}$$

The solution of (69) can be shown to be:

$$W(x) = A_1 \sin \beta x + A_2 \cos \beta x + A_3 \sinh \beta x + A_4 \cosh \beta x \tag{75}$$

where A_1, A_2, A_3 and A_4 are constants to be determined from the boundary conditions.

1.12 Example: Cantilever Beam

The boundary conditions for a cantilever beam are given by:

At $x = 0, \quad W = 0, \quad \dfrac{dW}{dx} = 0$

At $x = L, \quad \dfrac{d^2W}{dx^2} = 0, \quad \dfrac{d^3W}{dx^3} = 0$

Upon substitution into (70), and after some algebraic manipulation, we obtain the *eigenfunctions for a cantilever beam* as:

$$W_n(x) = A_n\left[\sin\beta_n x - \sinh\beta_n x - \frac{\sin\beta_n L + \sinh\beta_n L}{\cos\beta_n L + \cosh\beta_n L}\left(\cos\beta_n x - \cosh\beta_n x\right)\right] \qquad (76)$$

where β_n is obtained by solving

$$\cos\beta L \cosh\beta L = -1 \qquad (77)$$

Equation (72) can be solved numerically to give the *eigenvalues* $\beta_1 L, \beta_2 L, \ldots, \beta_n L$. The first three solutions can be shown to be:

$$\beta_1 = 1.8751/L$$
$$\beta_2 = 4.6941/L$$
$$\beta_3 = 7.8548/L$$

Now the natural frequencies can be obtained from $\omega_n = \beta_n^2\sqrt{\dfrac{EI}{\rho A}}$ and the first three values are obtained as:

$$\omega_1 = (1.8751)^2\sqrt{\frac{EI}{\rho A L^4}}$$
$$\omega_2 = (4.6941)^2\sqrt{\frac{EI}{\rho A L^4}}$$
$$\omega_3 = (7.8548)^2\sqrt{\frac{EI}{\rho A L^4}}$$

Figure 16 shows a typical frequency response plot of a cantilever beam, acted upon by a harmonic force at its tip. The amplitude of the tip motion is plotted as a function of the excitation frequency. The peaks occur at the *natural frequencies* of the system, and the deformation pattern of the beam (eigenfunctions) at each frequency is plotted at the top.

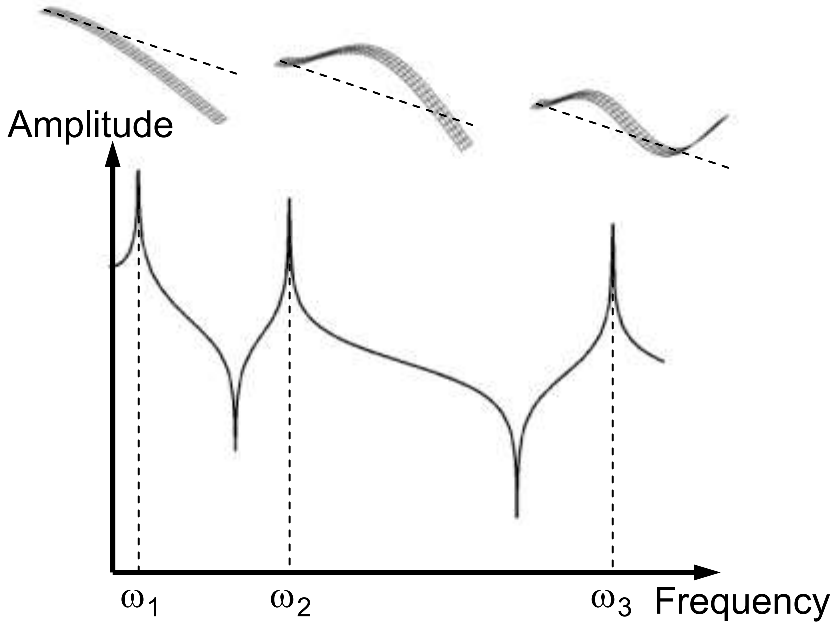


Figure 16. Frequency response of a beam.